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MULTIFRACTALITY OF PRIME NUMBERS

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The multifractal formalism is applied to prime numbers. The spectrum of critical indices is found to be contained in the interval $(\alpha_{\min}, 1)$, where $\alpha_{\min}$ tends to 1 for increasing sets of numbers. Besides the scaling of moments with respect to the length of intervals the scaling with respect to the sizes of subsets of natural numbers is also considered. We have found the cusps in the plots of the functions $f(\alpha)$ and we claim that they are not caused by numerical roundings but they are a real effect. Besides the computer method, some analytical calculations are presented.

1. Introduction

Recently there was a new method has been proposed for describing strange sets [1, 2], for review see ref. [3]. Since the pioneering paper [1] this method was applied to a variety of fractal phenomena, e.g. to diffusion limited aggregation (DLA) [4], the Hénon attractor [5], the Feigenbaum attractor [6] and Julia sets [7]; it was also applied to the Ryleigh–Bénard experiment at the onset of chaos [8].

The strange sets appearing in dynamical systems usually are not self-similar and due to this fact cannot be characerized by the fractal dimension alone. For a characterization of such sets Hentschel and Procaccia [9] introduced the family D_q of dimensions, where q is continuous. The authors of ref. [1] generalized further the concepts of ref. [9] and defined besides D_q the function $f(\alpha)$ allowing the recognition of the global universalities in dynamical systems. For example in ref. [8] it was shown that the experimentally observed orbit in a forced Ryleigh–Bénard experiment is in the same universality class as the orbit of the circle map.

The formalism developed in ref. [1] can be summarized for our purposes as follows: Let S be a set embedded in d-dimensional Euclidean space $\mathbb{R}^d$, where d is the smallest dimension which is sufficient for embodying S. Let S_1,

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$S_2, \ldots, S_{M(l)}$ denote the partitioning of S into $M(l)$ disjoint subsets such that each piece S_i lies within a sphere of radius l. Usually with the strange set there is associated some natural measure; let p be such a measure. Let us assume that the measure p is probablistic, i.e. $p(S) = 1$. Now we can define for a particular covering $\{S_i\}$ a partition function

$$\chi_q(l) = \sum_{i=1}^{M(l)} p^q(S_i), \tag{1}$$

where q is a real number and $p(S_i)$ is a measure of the set S_i. In examples studied so far it was found that the moments $\chi_q(l)$ behave like a power of length l,

$$\chi_q(l) \sim l^{\tau(q)}, \tag{2}$$

where the function $\tau(q)$ does not depend on l and characterizes the set S. For a given case relation (2) holds in the appropriate regime of l: for some sets it holds for $l \ll 1$ and for other, like e.g. for DLA, it holds for $l \gg 1$. The importance of the powerlike behaviour (2) was recognized earlier by P. Grassberger and I. Procaccia [10] for the particular value $q = 2$, i.e. for the so-called correlation function. If the set S is homogeneous (self-similar) and the measure p is proportional to the volume ("capacity") it can be easily shown that $\tau(q) = (q-1)D_0$, where D_0 is the usual fractal dimension. Owing to this fact it is natural to introduce the following generalized fractal dimensions:

$$D_q = \tau(q)/(q-1). \tag{3}$$

It can be proved that D_q is a decreasing function,

$$D_{q'} \leq D_q \qquad \text{for } q' > q.$$

If D_q is not constant the amount of difference $D_q - D_0$ gives a rough measure of the inhomogeneity of S. Further information on the structure of the supports of the measure p is supplied by means of the following change of variables. Let us introduce the function $\alpha(q)$ measuring the slope of $\tau(q)$:

$$\alpha(q) = \frac{\mathrm{d}\tau(q)}{\mathrm{d}q}. \tag{4}$$

It can be shown that $\tau(q)$ is concave and because of this fact the above relation can be inverted – so instead of q the independent variable is now α. Next instead of $\tau(q)$ the following function is introduced:

$$f(\alpha) = \alpha q(\alpha) - \tau(q(\alpha)) , \tag{5}$$

where q is expressed by α via relation (4). Roughly speaking the variable α tells how the measure of the subset S_i depends on the length l,

$$p_i \equiv p(S_i) = l^{\alpha_i} , \tag{6}$$

where the "critical" exponent α_i takes values between $\alpha_{\min}$ and $\alpha_{\max}$. It follows from (3)–(5) that

$$\alpha_{\min} = D_{\infty} , \qquad \alpha_{\max} = D_{-\infty} . \tag{7}$$

The function $f(\alpha)$ tells us what is the measure of the subset of S where the measure scales with exponent α. For self-similar fractal sets the spectrum of exponents consists of one point $\tilde{\alpha} = D_0$ and the corresponding value of function f is $f(\tilde{\alpha}) = D_0$. Sets possessing a nontrivial spectrum of α's (which corresponds to nonconstant D_q) are called multifractal sets. Let us stress that the f–α formalism is in fact connected with the *supports* of the measure p and not with the set S; sometimes the terms: multifractal sets and multifractal measures, are used synonymously.

Above we have considered one particular covering $\{S_i\}$ of S and others coverings will in general lead to different $f(\alpha)$. To avoid the dependence on the covers the infimum of the partition function over all coverings is taken [1, 18]. In following sections we will assume that there is no significant dependence on the coverings chosen and we will use the simplified version of the formalism with one particular partition.

Originally the above method was invented for dynamical systems, but it can be applied to any set. In this paper we are going to look for the multifractal properties of the set of prime numbers. In contrast to the most cases studied so far by the multifractal formalism, the set of prime numbers allows some analytical estimates of the moments χ_q and the information about the scaling regimes can be obtained without use of the computer. Also the form of the deviation from the powerlike behaviour (2) can be obtained analytically. We think that although the prime numbers are unphysical it is reasonable to test the multifractal formalism on a set less trivial than e.g. the Cantor sets, which allows strict evaluation of the moments χ_q. In the next section we will present the results of computer calculations for some finite subsets of natural numbers. In section 3 some analytical estimation are presented. Finally section 4 contains concluding remarks and conjectures.

2. Computer calculations

For the purpose of simplifying the programs we were considering subsets of the natural numbers of the form $S(\nu) = \{1, 2, \ldots, N = 2^\nu\}$. The customary notation for a function giving the number of prime numbers smaller or equal to x is $\pi(x)$. Next we define the measure of the subset $S(n) = \{i_1, i_2, \ldots, i_n\} \in S(\nu)$ as the number of prime numbers contained in it divided by $\pi(2^\nu)$. Let us split the set S into the intervals $I_n(k)$ of length $l = 2^k$ of the form

$$\begin{aligned} I_n(k) &= \{(n-1)2^k + 1\,, \quad (n-1)2^k + 2\,, \quad \ldots, \quad (n-1)2^k + 2^k\} \\ &= \{(n-1)l + 1\,, \quad (n-1)l + 2\,, \quad \ldots., \quad nl\}\,, \end{aligned}$$

where $n = 1, 2, 3, \ldots, M(l) = 2^{\nu - k}$, $k_{\min} \leqslant k \leqslant k_{\max}$ and this range depends on the value of ν. Now we can write

$$S = I_1(k) \cup I(k)_2 \cup \ldots. \cup I_{M(l)}(k)\,.$$

The measure of the interval $I_n(k)$ is given by

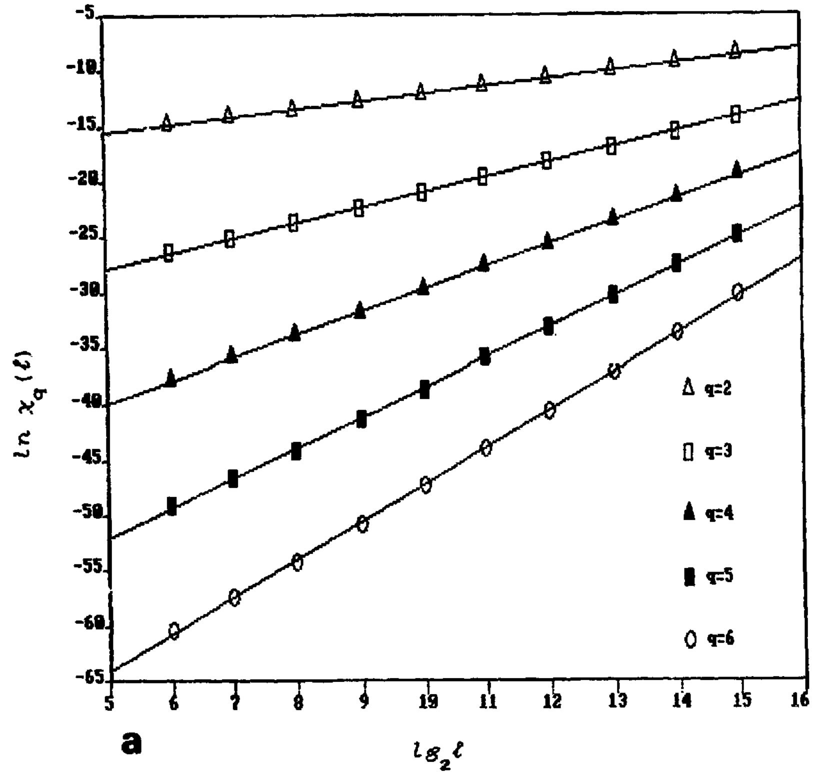
$$p(I_n(k)) = \frac{\pi(n2^k) - \pi((n-1)2^k)}{\pi(2^\nu)} \tag{8}$$

and moments of this measure are given by

$$\chi_q^N(l) = \sum_{i=1}^{N/l} p^q(I_i(k))\,. \tag{9}$$

From this definition it follows that $\chi_1(l) = 1$. The sets I_n are ordinary intervals and of course are not interesting in itself but the measure defined on them by (8) possess, as we will see further, quite nontrivial properites.

We have performed computer calculations of the above quantities $p(I_n)$ for values of $\nu = 18$, 19, 20, 21, 22, and 23, which corresponds to the range of natural numbers searched from $N = 262\,144$ to $N = 8\,388\,608$. Each time the values of k were chosen between the values $k_{\min} = \nu - 12$ and $k_{\max} = \nu - 3$; it means that the number of intervals needed to cover S was 4096 for the most subtle refinement and was 8 in the case of the largest intervals. The computer calculated the quantities $p(I_n)$ given by (8) exactly and next the moments $\chi_q(l)$ were calculated. The parameter q was changed in the interval $-20 \ldots 100$ with step 0.25. Figs. 1 and 2 shows the plot of $\ln \chi_q(l)$ vs $\ln l$ for a few values of q and for $\nu = 18$ and $\nu = 23$ respectively. The functions $\tau(q)$ were calculated by means of the least squares method. In the worst cases the standard deviation



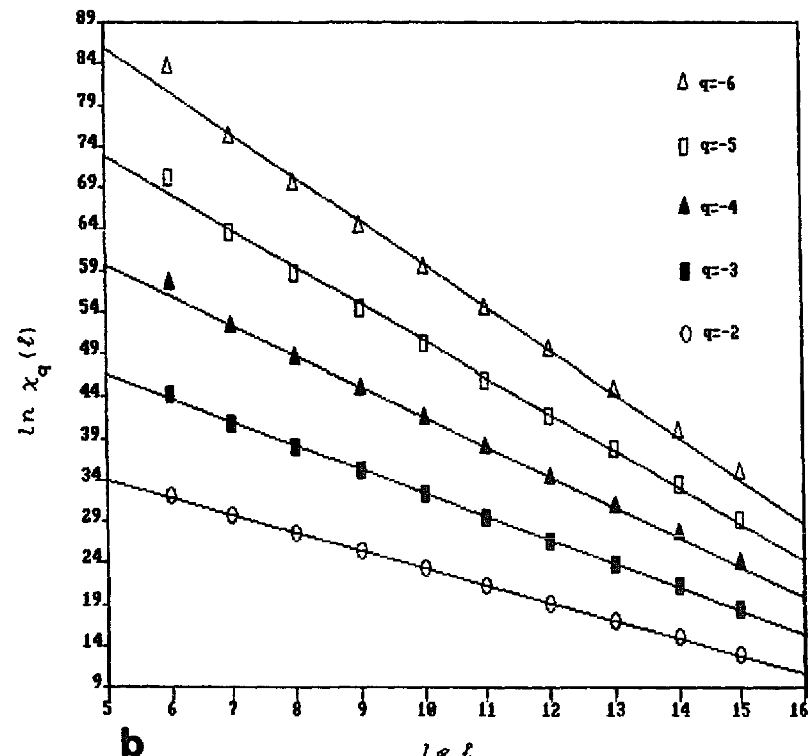


Fig. 1. (a, b, c) The plot of $\ln \chi_q(l)$ vs $\lg_2 l$ for $\nu = 18$ and different q's. The straight lines are the least squares fits to the points. These straight lines intersect for $l = \nu$ (=18).

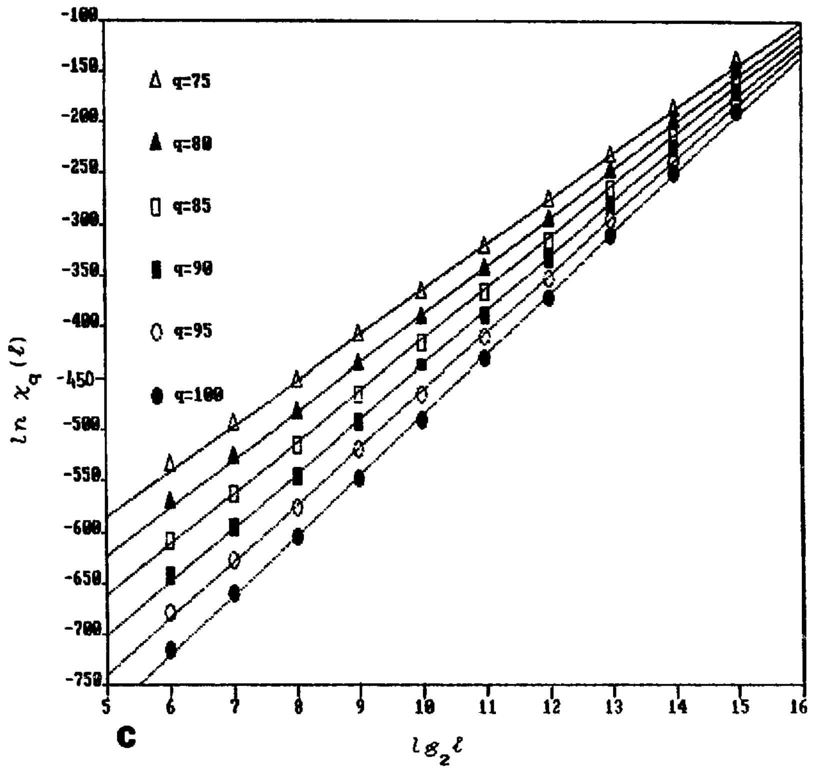


Fig. 1 (cont.).

from the powerlike dependence was about 1–2 percent, so we conclude that in the investigated regimes of l and q the scaling law (2) for prime numbers is fulfilled with high accuracy. Finally we have calculated numerically the functions $\alpha(q)$ and $f(\alpha)$. The resulting functions $f(\alpha)$ are shown in fig. 3. For each ν the maximal value of α $(=D_{-\infty})$ and the minimal value of α $(=D_{\infty})$ shifts towards $\alpha = 1$; we will discuss this phenomenon in the next section, were we will show analytically that $\alpha(-\infty) = 1$ and $f(\alpha(-\infty)) = 1$. We have found the same changes of D_{100} with the variation of ν; see the plot of D_q vs q shown in fig. 4. We see from these figures that D_q are very close to $D_{-\infty}$ for q just below zero, but D_q are far from D_{∞} even for $q = 50$.

From fig. 3 it is seen that the functions $f(\alpha)$ have the cusps near the value of $\alpha_{\min}$; the multivaluedness of the function $f(\alpha)$ is caused by the fact that the computer has plotted in fact the pairs $\{\alpha(q), f(q)\}$ and not the function $f(\alpha)$ alone. The possible reasons of the origin of such a cusp will be discussed later on in this section. The values of f for α near $\alpha_{\min}$ are negative and remain such in the neighbourhood of zero, but the values of $f(\alpha_{\max})$ shift to 1 with increasing ν. The sets with negative fractal dimension are called "volatile" [11].

As remarked by Coniglio [12] the sets which are in principle unbounded can

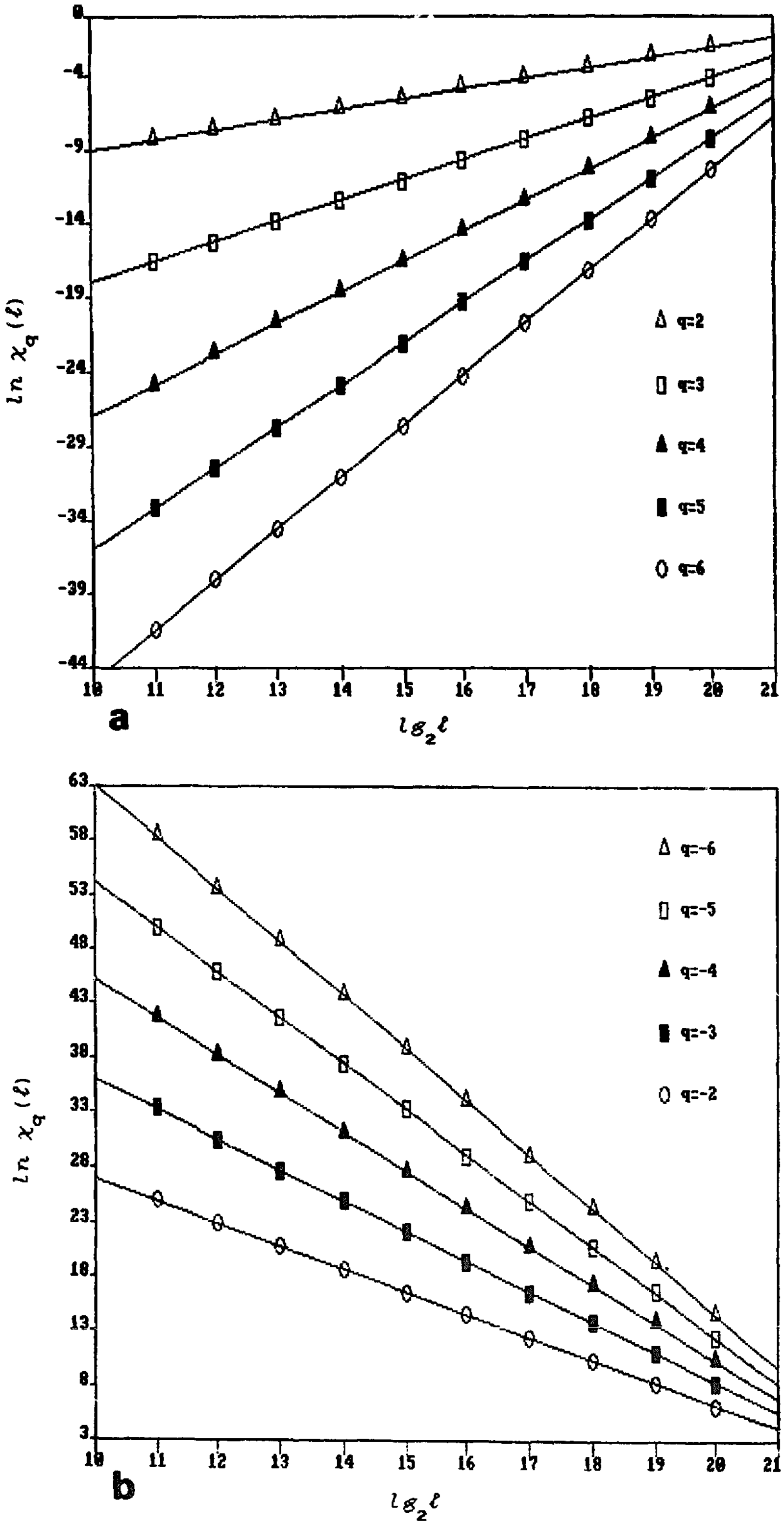


Fig. 2. The same as in fig. 1 but for $\nu = 23$.

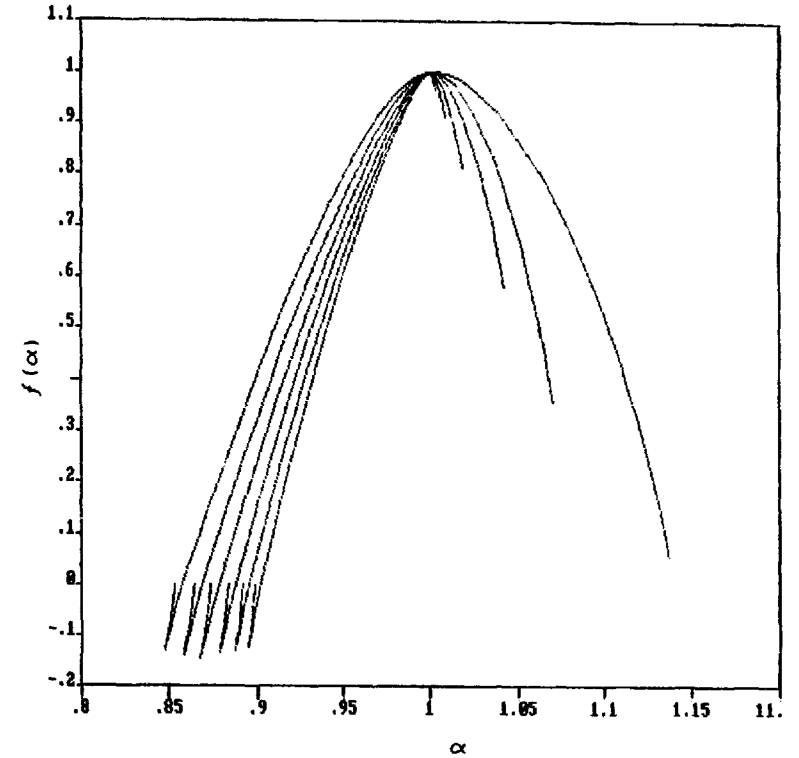


Fig. 3. The plots of $f(\alpha)$ for $\nu = 18$ to 23 from left to right respectively.

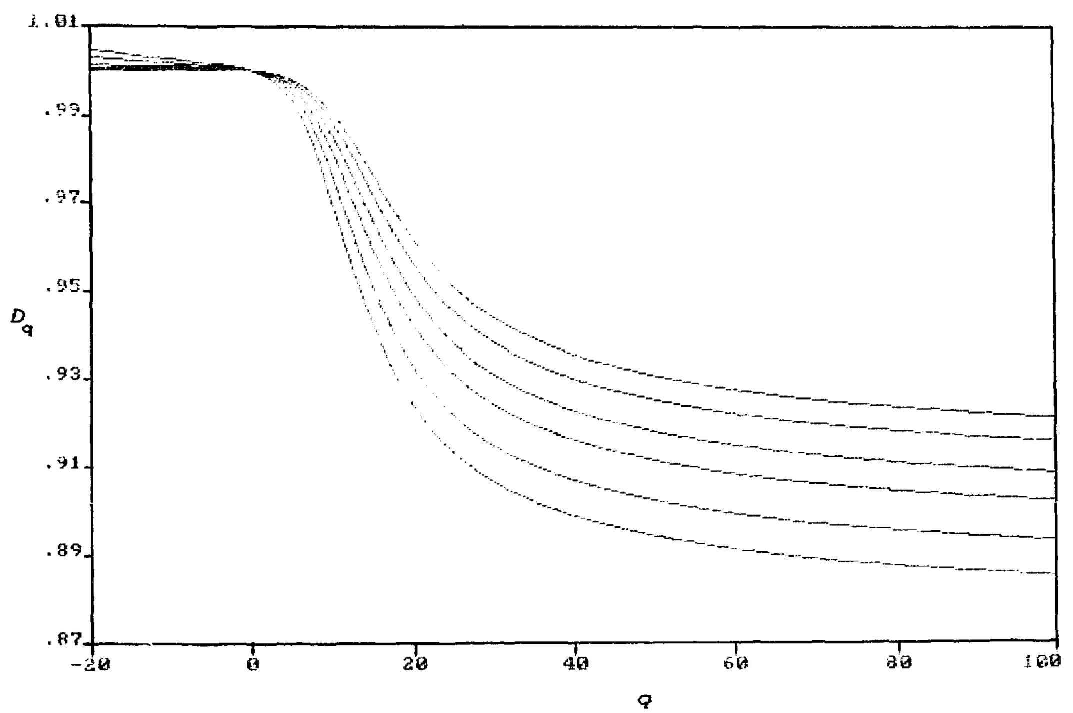


Fig. 4. The dependence of D_q on q for $\nu = 18$ to 23 in upward direction.

display a scaling of the moments with respect to the linear sizes of the system – for example aggregates can grow up to arbitrary diameter L. In our case the moments χ_q provide an example of the scaling with sizes N of the intervals $\{1, \ldots, N\}$. Namely we can cover different subsets of the natural numbers by intervals of the same length l and look for a dependence on N of the appropriate moments. In other words we keep l fixed in the definition (9) and obtain $\chi^l_q(N)$ which corresponds to the definition of the fractal dimension via the relation between the linear size and the "mass" of the fractal.

The common values of l for different N's were in the range 11 . . . 15 and we obtained five families of moments $\chi^l_q(N)$ and five functions $f(\alpha)$. The plot of $\ln \chi^l_q(N)$ vs $\ln N$ for few values of q is shown in fig. 5 for $l = 11$ and $l = 15$. Also for other l we have found the powerlike dependence:

$$\chi^l_q(N) \sim N^{-\tau(q)} .$$

The functions $\tau(q)$ again were determined by the least squares method now from six points corresponding to $N = 18 \ldots 23$ (although only two points are sufficient to determine the straight line). The resulting functions $f(\alpha)$ are shown in fig. 6a. As we see now the functions $f(\alpha)$ have a cusps in the neighbourhood of $\alpha_{\max}$. The magnification of this region is shown in fig. 6b. Initially we suspected that it was a computer artefact. We tried to use the step for the numerical differential of $\tau(q)$ in the range 0.025 . . . 1.0; we have also changed other parts of the programs – the structure was absolutely persistent to any such modifications. The only explanation of the turnbacks of $f(\alpha)$ in figs. 3 and 6 are the oscillations in the slopes of log–log plots of the moments. Such oscillations were previously reported in the literature [13, 14, 5] and they are inherent to the lacunar fractal sets [11]. These oscillations manifest at larger order moments – for q's in the intermediate region the overall changes in $\alpha(q)$ (or equivalently D_q) are dominant or the deviations from the powerlike behaviour are absent for such q's. The deviations from the straight lines in figs. 1, 2 and 5 are practically invisible; see however the slight oscillations around a straight line for $q = 95$. We have plotted $\alpha(q)$ for the case of scaling with N in fig. 7 and contrary to the general prediction they are not monotonic. The curves for $l = 13$, 14 and 15 have an extremum in one point and $\alpha(q)$ for $l = 11$ and 12 has extrema in two points corresponding to two cusps in fig. 5b. We have checked that up to $q = -100$ there is no further oscillation and $\alpha(q)$ reaches a plateau. A further justification for such an explanation is given in fig. 8 were the difference in the shape of function $f(\alpha)$ caused by the different number of points taken for the least squares method are shown. In fig. 8 the functions f are plotted for a case of scaling with respect to l for $N = 18$; for other N or f's for a scaling with respect to l we have not observed such

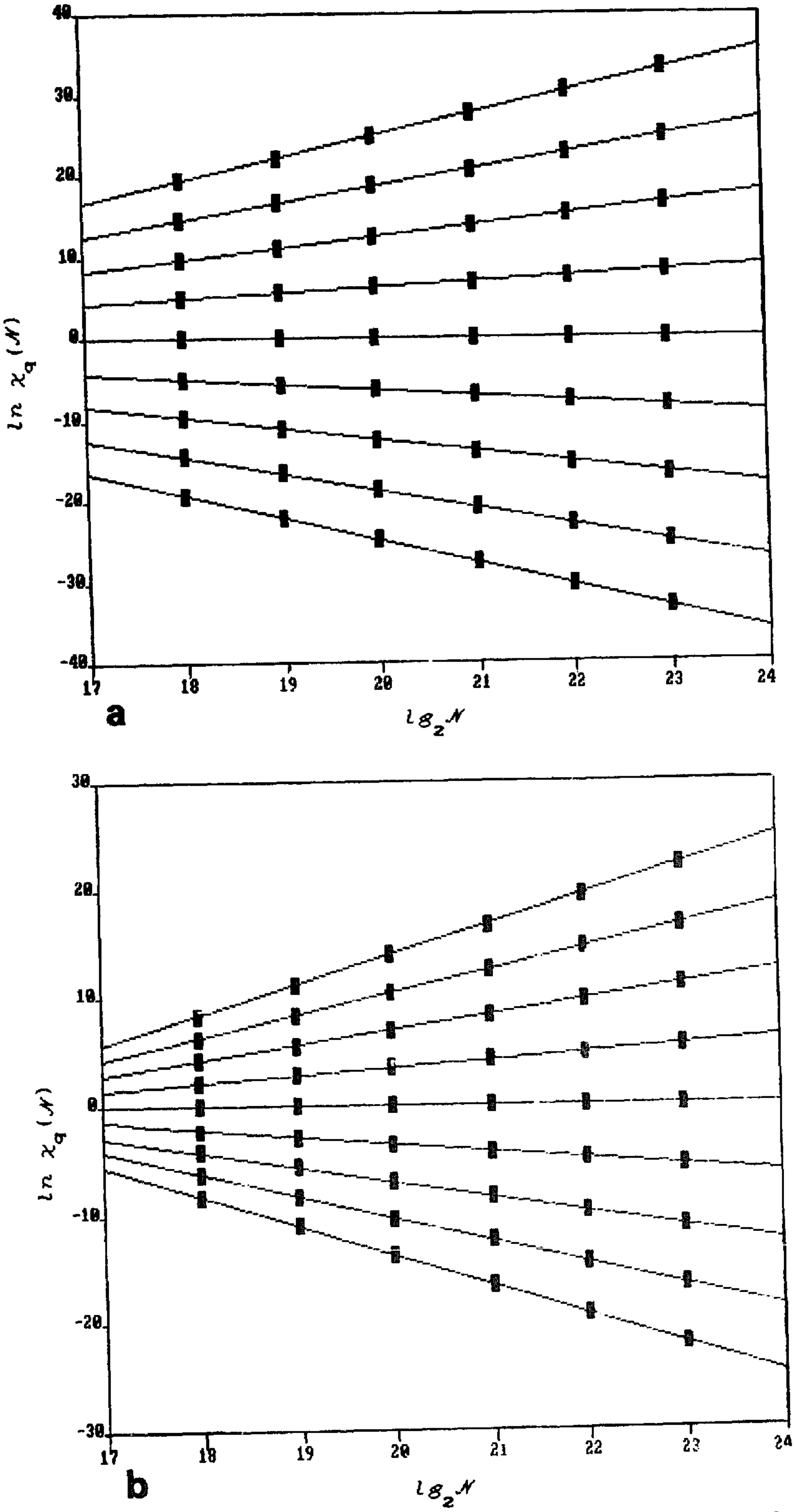


Fig. 5. The plot of $\ln \chi_q(l)$ vs $\lg_2 N$ for $l = 11$ (a) and $l = 15$ (b) for $q = -4, -3, \ldots, 4$.

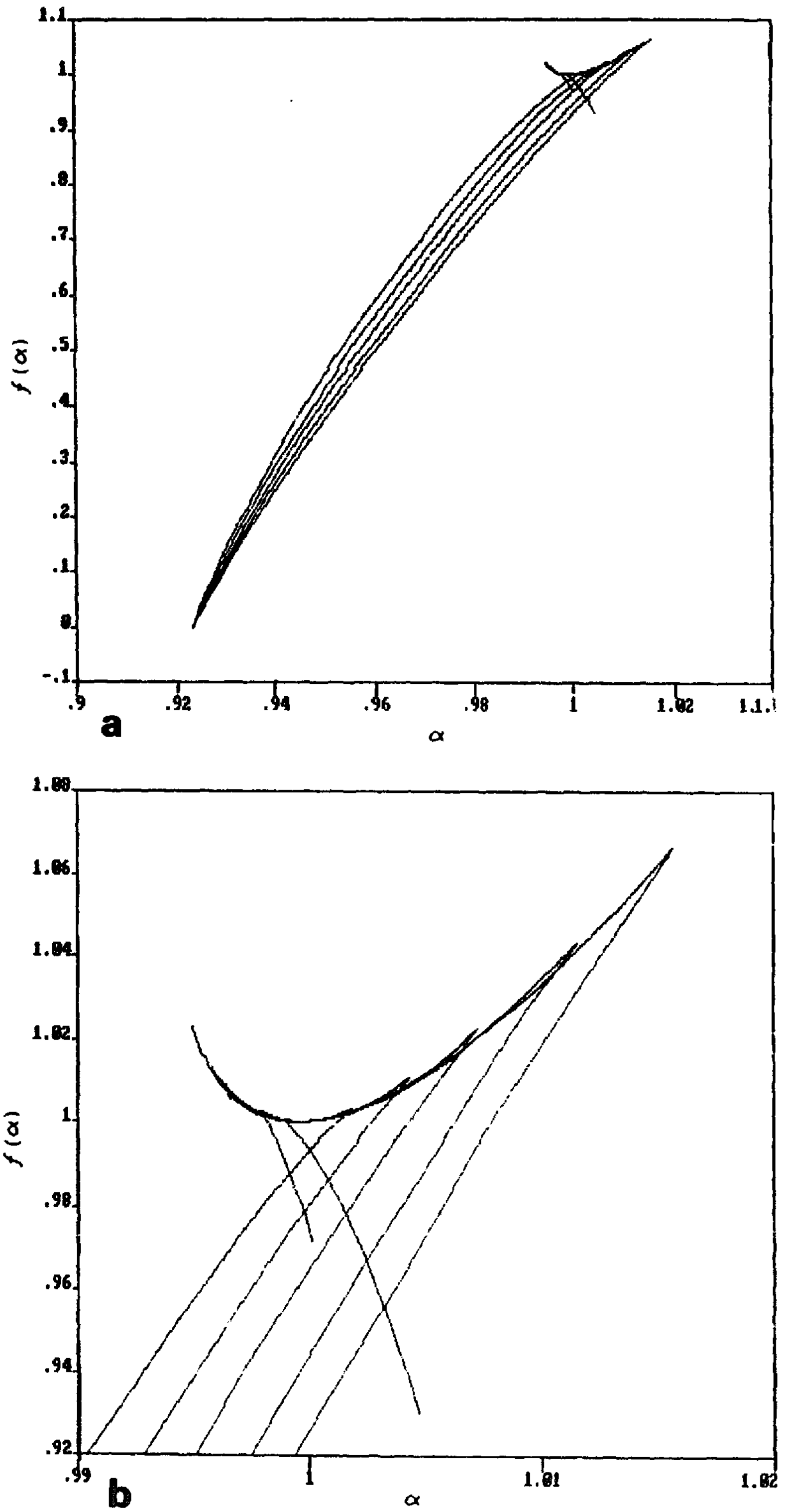


Fig. 6. (a) The plots of $f(\alpha)$ for $l = 11$ to 15 from right to left respectively. The magnification of the cusps are shown in (b).

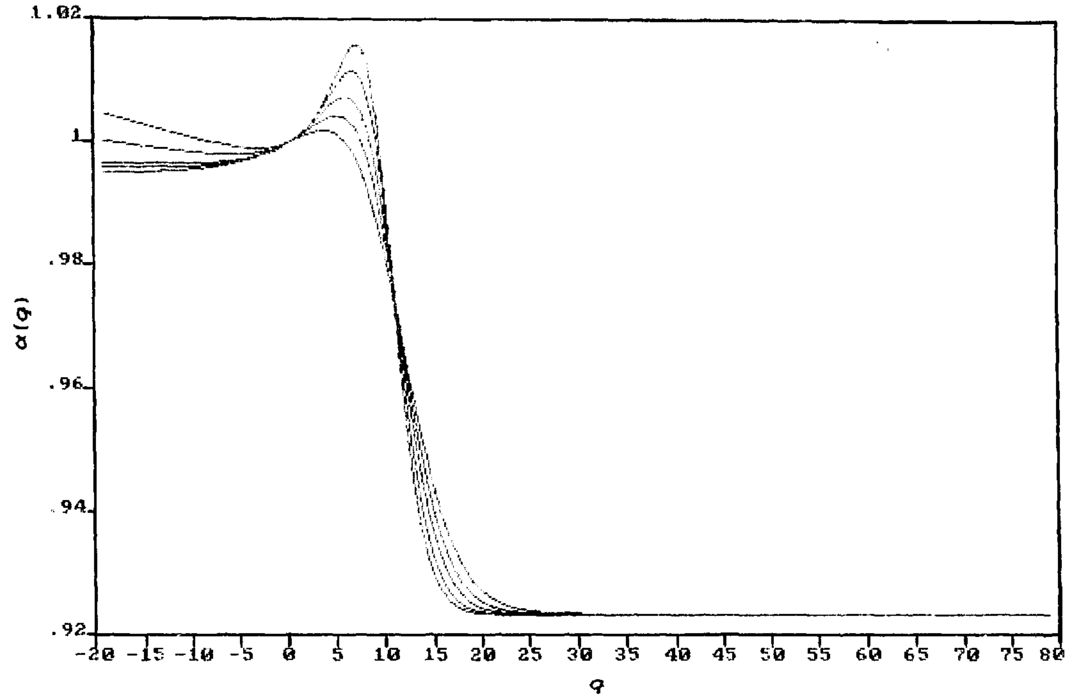


Fig. 7. The dependence of $\alpha(q)$ for different l's.

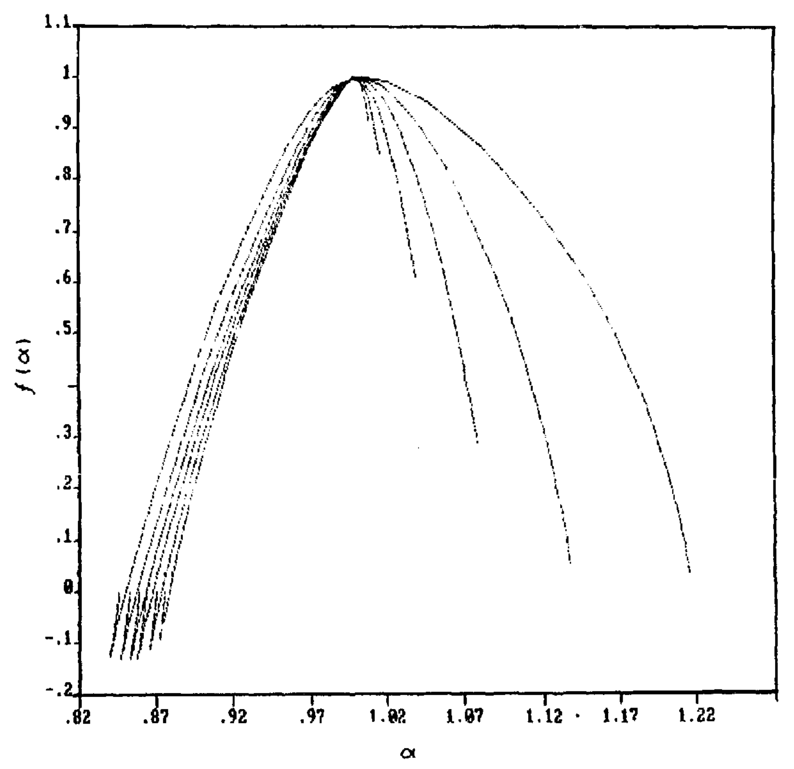


Fig. 8. The plots of $f(\alpha)$ for $\nu = 18$ and decreasing number of l's taken for the least squares method from right to left respectively.

significant differences. To determine all curves in fig. 3 we used 9 values of l; we skip the smallest l because as we see, from fig. 1b, $\ln \chi$ deviates the most from the straight lines just for the shortest interval. A similar effect was observed by Arneado et al. [5] for the Hénon attractor.

L.A. Smith et al. [14] linked oscillations to the logarithmic corrections to the powerlike dependence of moments:

$$\ln \chi^q = \tau(q) \ln l + \psi(\ln l) + \ldots ,$$

where ψ is a periodic function. We think that the oscillating character of deviations from the straight lines as well as the periodicity of ψ are the property of "true" fractal sets and it is not a case for prime numbers: we will show in the next section for large $|q|$ the existence of terms of the form $\ln \ln l$ and $\ln \ln N$ which are not periodic.

As can be remarked from fig. 5 the reflection of curves in the line $\alpha = \alpha^*$, α^* is the turning-point, and next in the line $f = f^*$, here $f^* = f(\alpha^*)$, changes the functions $f(\alpha)$ into smooth curves – the result of these operations is shown in fig. 9. At first sight it looks like a mystery but this kind of symmetry with respect to the reflection is caused by the fact that $\alpha(q)$ changes continu-

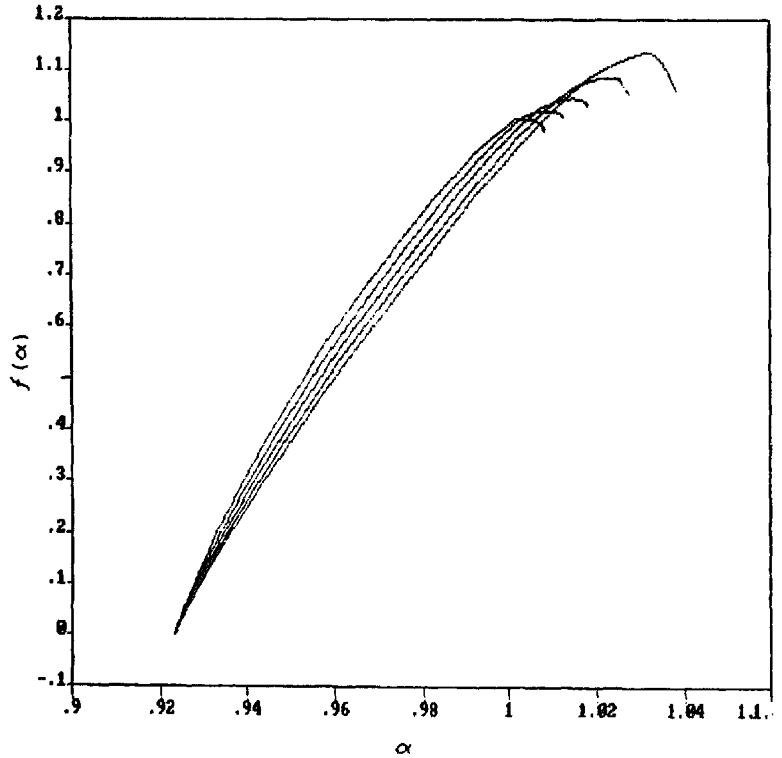


Fig. 9. The "rectification" of the plots in fig. 6.

ously at the points α^*. Also Legendre transformations applied "by force" to other functions which are not convex or concave (like e.g. $\cos x$ for $x \in (0, \pi)$) will possess this property of "rectification". But a kind of mystery is the meeting of funtions $f(\alpha)$ for all l's in one point: at $\alpha = 0.9233\ldots$ all f's are equal to zero. We have no explanation of this curious fact.

In the next section we will present the analytical calculations of moments $\chi_q(l)$ and the spectrum of α.

3. Analytical calculations

Motto: Truth is too complicated to allow anything but approximations

John von Neumann

In this section we are going to understand the results presented above and obtained with the help of the computer. The idea is to estimate the measure $p(I_n(l))$ given by (8). In this section we will consider a subset $\{1, 2, \ldots, N\}$ of $\mathbb{N}$ which we will cover by disjoint intervals of length l, where $N/l \in \mathbb{N}$; the particular choice of N and l in the previous section in the form of the powers of two was made for the sake of convenience.

Let us recall that there exists a lot of formulas expressing the function $\pi(x)$ in terms of other funtions. The best estimation, valid under the assumption of the Riemann hypothesis on zeros of the zeta function $\zeta(s)$, is the following one (see e.g. ref [15], formulas (5.1.50)):

$$\pi(x) = \mathrm{li}(x) + \mathcal{O}(\sqrt{x}\ln(x)), \tag{10}$$

where the logarithmic integral is given by

$$\mathrm{li}(x) = vp\int_0^x \frac{1}{\ln(u)}\,\mathrm{d}u. \tag{11}$$

The logarithmic integral has the following asymptotic expansion:

$$\mathrm{li}(x) = x\left(\frac{1!}{\ln(x)} + \frac{2!}{(\ln(x))^2} + \ldots + \frac{r!}{(\ln(x))^r} + \ldots\right). \tag{12}$$

Keeping the first term only we obtain for $x \gg 1$ the following approximation:

$$\pi(x) = \frac{x}{\ln(x)}. \tag{13}$$

The above formula was guessed by sixteen-year-old Gauss in 1793, see ref. [16].

From (13) it follows that prime numbers do not form the self-similar set as the "mass" of the interval of length L is given by $L/\ln L$ and this quantity does not display the rescaling "covariance".

Substituting (13) into the definition of the measure $p(I_i(l))$ we obtain

$$p(I_i(l)) = \left(\frac{(i+1)l}{\ln((i+1)l)} - \frac{il}{\ln(il)}\right) N/\ln(N) .$$

For large l we have

$$\ln((i+1)l) \cong \ln(il) \tag{14}$$

and the above formula can be written as

$$p(I_i(l)) = \frac{1}{\pi(N)} \frac{l}{\ln(il)} . \tag{15}$$

Putting the above expression into (1) we obtain (now $M(l) = N/l$)

$$\chi_q(l) = \pi(N)^{-q} \sum_{i=1}^{N/l} \frac{l^q}{(\ln(il))^q} . \tag{16}$$

Because $\ln x$ changes slowly for large x we can approximate the sum by an integral:

$$\chi_q(l) = \pi(N)^{-q} l^{q-1} \int_l^N \frac{\mathrm{d}u}{(\ln(u))^q} . \tag{17}$$

As the verification of our approximations let us look for the partition function for $q = 1$; it should be equal to one. For $q = 1$ we obtain from (17) in view of (10) and (11)

$$\chi_1(l) = \pi(N)^{-1}(\pi(N) - \pi(l))$$

and for $N \gg l$ we have $\chi_1(l) = 1$.

Let us change the variables in the integral appearing in (17) and let us denote the resulting integral by $I(q)$:

$$\chi_q = \pi(N)^{-q} l^{q-1} (-1)^{q-1} I(q) , \tag{18}$$

where

$$I(q)=\int_{-\ln l}^{-\ln N} t^{-q}\,\mathrm{e}^{-t}\,\mathrm{d}t\,. \tag{19}$$

For $q<1$ this integral can be expressed by the incomplete gamma function $\gamma(a,x)$ ([15], formula 6.5.2):

$$\gamma(a,x)=\int_0^1 \mathrm{e}^{-t}\,t^{a-1}\,\mathrm{d}t\,,\qquad a>0\,.$$

We obtain

$$I(q)=\gamma(1-q,-\ln N)-\gamma(1-q,-\ln l)\,.$$

Using the asymptotic expansion of $\gamma(a,x)$ we obtain for $\ln N \gg \ln l \gg 1$ and fixed q satisfying $-q \ll \ln l$ (remember that $q<1$!) the estimation

$$\chi_q=\left(\frac{l}{N}\right)^{q-1}\,. \tag{20}$$

From the above it follows that (in the appropriate regime of l's and q's) we have a scaling with $\tau(q)=q-1$ and $D_q=1$ for q just below 1; this last result is confirmed by the plot of D_q in fig. 4. The next check of the formula (20) is given in the table I; it shows a sample of values of $\chi_q(l)$ calculated exactly by means of the brute force method described in the previous section and via (20).

For $q>1$ the integral (19) cannot be evaluated explicitly. For $q=n$, n integer, after n integrations by part we obtain for the integral in (17)

$$\int\frac{\mathrm{d}u}{(\ln u)^n}=-u\left(\frac{1!}{(n-1)(\ln u)^{n-1}}+\ldots+\frac{1}{(n-1)!\ln u}\right)+\frac{1}{(n-1)!}\,\mathrm{li}\,(u)\,,$$

so we end up with the logarithmic integral. Making use of the asymptotic expansion (12) leads to the exact cancelation of n terms from series (12) with terms integrated out by parts and we obtain the estimation (20). As is seen from the table I the estimation (20) is wrong for large q. The reason for it is that after the cancelation of n terms of the series expansion (12) with the terms integrated out by parts we are left with the nth term of series (12). But starting with the nth term, where $n\sim\ln x$, the terms of asymptotic expansion (12) are divergent. For $l=16\,384$ the optimal number of terms in expansion (12) is 9, and for $q\geqslant 10$ the formula (20) is not valid – it agrees with table I. Let us

Table I
$N = 2\,097\,152$, $l = 16\,384$.

q	$\left(\frac{l}{N}\right)^{q-1}$	$\chi_q(l)$
−40.0	2.48662E + 0086	1.71672E + 0087
−35.0	7.23701E + 0075	3.58277E + 0076
−30.0	2.10625E + 0065	7.59661E + 0065
−25.0	6.12998E + 0054	1.64058E + 0055
−20.0	1.78406E + 0044	3.62141E + 0044
−15.0	5.19230E + 0033	8.21334E + 0033
−10.0	1.51116E + 0023	1.92988E + 0023
−5.0	4.39805E + 0012	4.76835E + 0012
0.0	1.28000E + 0022	1.28000E + 0002
5.0	3.72529E − 0009	4.12082E − 0009
10.0	1.08420E − 0019	2.18450E − 0019
15.0	3.15544E − 0030	2.73789E − 0029
20.0	9.18355E − 0041	5.93800E − 0039
25.0	2.67276E − 0051	1.51644E − 0048
30.0	7.77877E − 0062	4.03799E − 0058
35.0	2.26392E − 0072	1.08850E − 0067
40.0	6.58887E − 0083	2.94638E − 0077

mention that in deriving the formula (20) we used only the first term of the asymptotic expansion for the incomplete gamma function.

Now we estimate the integral (19) for large $|q|$. Let us rewrite it in the form

$$I(q) = (-1)^{q-1} \int_{\ln l}^{\ln N} e^{t - q\ln(t)}\, dt \,.$$

For sufficiently large q we can write

$$t - q\ln(t) \approx -q\ln(t) \,.$$

The above approximation is justified for $|q] \gg \ln N$. Then we obtain

$$I(q) = \frac{(-1)^{q-1}}{1-q}\left((\ln N)^{-q+1} - (\ln l)^{-q+1}\right).$$

For q negative the contribution from the upper limit of integration is relevant and we obtain

$$\chi_q(l) = \frac{1}{1-q}\,\frac{\ln N}{N}\left(\frac{l}{N}\right)^{q-1}. \tag{21}$$

On the other hand for q positive the contribution from the lower limit of integration is relevant and we have

$$\chi_q(l) = \frac{1}{q-1}\left(\frac{\ln N}{N}\right)^q \left(\frac{l}{\ln l}\right)^{q-1} . \tag{22}$$

From (21) it follows that the scaling law (2) with respect to l is fulfilled with $\tau(q) = q - 1$ for $-q \gg 1$, and from that it follows that $D_{-\infty} = \alpha_{\max} = 1$ and consequently $f(\alpha_{\max}) = 1$. This conclusion agrees perfectly with the results obtained from explicit computer calculations. The falling down of function $f(\alpha)$ comes from $\alpha(q)$ for q larger than $\ln l$ and smaller than $\ln N$. From (21) we see that for negative q the simple scaling of moments with respect to N in the form of powers of N is violated and the logarithmic corrections are present. Because of that the function $\tau(q)$ is no longer concave, but this is insufficient as an explanation of cusps of the functions $f(\alpha)$ near $\alpha_{\max}$ because the turnbacks in fig. 6 were detected already for $q = -20$ what is outside the range of applicability of (21).

We see from (22) that for $q \gg \ln N$ the powerlike behaviour (2) of the moments with respect to l and N is violated and the value of $\alpha(\infty)$ is not determined.

4. Conclusions

We have shown for subsets $\{1, \ldots, N\} \in \mathbb{N}$, where largest N was equal to 8 388 608, that the moments $\chi_q^N(l)$ and $\chi_q^l(N)$ have the powerlike dependence (2) which holds with very high accuracy for appropriate regimes of the orders q. For a scaling with respect to l and N the functions $f(\alpha)$ do not obey the equality $f(\alpha_{\min}) = f(\alpha_{\max})$. The generalized dimensions D_q take values from the interval $(\alpha_{\min}, 1)$ and we conjecture that the value $D_\infty = \alpha_{\min}$ converges to one for $N \to \infty$ for the case of scaling with respect to l. The example of prime numbers shows explicitly that for sufficiently large q's there can be departures from the powerlike dependence. In connection with that let us remark that to our knowledge the multifractality for DLA was investigated for moments of order $q = -4 \ldots 8$ and it is possible that for larger q the situation can change. It is possible that reported recently [17] deviations from scaling for extremely large aggregates are caused by such a mechanism.

We have found explicitly the logarithmic corrections to the simple hypothesis (2). We suggest that for such a case it can be reasonable to change the usual f–α formalism by introducing the second besides $\tau(q)$ function $\sigma(q)$ by means of the definition

$$\chi_q(l) \sim l^{\tau(q)}(\ln l)^{\sigma(q)} .$$

Of course for such a case the functions τ and σ need not be longer concave and because of that it will not be possible to perform the Legendre transformation and obtain the function $f(\alpha)$. It seems to suggest that families of critical exponents $\tau(q)$ are more fundamental objects than $f(\alpha)$.

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