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# MULTIFRACTALITY OF RANDOM PRODUCTS

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Moments of the measure defined by products of random deviates are considered with the aim of reproducing the multifractal spectrum of the DLA.

In 1981 WITTEN and SANDER have proposed the Diffusion-Limited Aggregation (DLA) model [1] leading to the formation of fractal patterns. The growth process of DLA is governed by the set of probabilities $p_i$ describing hitting of random walks by the perimeter of a cluster. The multifractal formalism [2] was applied for the description of DLA in [3]. Multifractal description of DLA is based on the behaviour of moments of the measure given by $\{p_i\}_{i=1}^N$

(1) $$Z_q(N) = \sum_{i=1}^{N} p_i^q ,$$

where $N$ is the number of perimeter sites (linked to the linear size $R$ and mass $M$ of the DLA cluster by power-like relations) and $q$ is a real number. The quantity $Z_q$ is also called partition function. If these moments display a power-like dependence on $N$:

(2) $$Z_q(N) \sim N^{-\tau(q)}$$

and the function $\tau(q)$ is *nonlinear*, then the measure $p_i$ is called multifractal. The multifractality is described by means of the function $f(\alpha)$, which in turn is the Legendre transform of $\tau(q)$ with respect to the variable $q$:

(3) $$\alpha(q) = \frac{d\tau}{dq}, \qquad f(\alpha) = q\alpha(q) - \tau(q).$$

The simulations of DLA clusters [4, 5, 7] have revealed that for negative values of $q$ there is a breakdown of scaling and the moments display the following behaviour:

(4) $$Z_q(N) \sim \begin{cases} e^{-\beta q (\ln N)^{2.15}} & \text{for} \quad q < 0, \\ N^{-\tau(q)} & \text{for} \quad q > 0. \end{cases}$$

The change of power-like behaviour of moments for $q < 0$ is named "phase transition".

My aim here is to develop an algorithm reproducing the moments and $f(\alpha)$ with the properties of those of DLA, what should give some insight into the mechanism underlying the formation of DLA patterns.

As a model of the set of probabilities $\{p_i\}_{i=1}^N$ I am going to investigate here the products of random numbers $r$ of a given distribution (e.g. uniform, Poisson, Gamma deviates). So, if $p_i'$ denotes the product of the same number $k$ of deviates:

$$p_i' = r_1 \times r_2 \times \ldots \times r_k\,, \qquad i = 1, 2, \ldots, N, \tag{5}$$

then the normalized measure is obtained from the equation

$$p_i = \frac{p_i'}{\sum_{l=1}^N p_l'}\,, \qquad i = 1, 2, \ldots, N. \tag{6}$$

It can be easily shown using the Strong Law of Large Numbers that for large $N$ moments of such a measure are given by the equation:

$$Z_q(N) = \sum_{i=1}^N p_i^q \approx \frac{a_q}{(a_1)^q} N^{1-q}, \tag{7}$$

where $a_q = \langle p_i'^q \rangle$ is the average of the $q$-th power of $p_i'$. Because $\tau(q) = q - 1$ is a linear function, no multifractal behaviour is obtained – the spectrum $f(\alpha)$ is just one point ($\alpha = 1$, $f(\alpha) = 1$) in such a case. In particular, for $p_i$ being the product of $k$ random variables uniformly distributed on the interval $(0, 1)$ it follows that:

$$Z_q(N) \approx \begin{cases} \infty & \text{for} \quad q \le -1, \\ \dfrac{2^{qk}}{(1+q)^k} N^{1-q} & \text{for} \quad q > -1, \end{cases} \tag{8}$$

so for $q = -1$ there appears a phase transition, while for DLA the phase transition occurs at $q = 0$, see (4). The results (7) and (8) are trivial in the sense that there is no multifractal behaviour, but they are interesting because they belong to these rare cases when the moments can be calculated analytically.

The only way to obtain the nontrivial behaviour of $\tau(q)$ is to violate the property that $a_q = \langle p_i'^q \rangle$ is independent of $i$, what can be accomplished, for example, by taking random number $k$ of terms in the definition (5). In this case I was not able to prove any analytical formula like (7) or (8) and only the numerical results have been obtained. I have generated $N = 1000, 2000, \ldots, 16000$ numbers $p_i$ by taking products of deviates uniformly distributed on the interval $(0, 1)$, where the number of terms was chosen randomly between 2 and $3 \times m$, where $m$ denotes the stage of the "growth process" (i.e. $m = 1$ for $N = 1000$, $m = 2$ for $N = 2000$, ..., $m = 5$ for $N = 16000$). These runs were performed 500 times to obtain a large statistics and the averaged results are shown in the Figs. 1 and 2. The plots presented in these figures display some resemblance with $f(\alpha)$ obtained for DLA.

Another way of getting nontrivial $\tau(q)$ consists in introducing some kind of the "density of states" function $\mathcal{D}(i)$ and defining the moments by

$$Z_q(N) = \sum_{i=1}^N \mathcal{D}(i) a_q^{(i)}, \qquad \text{where} \quad a_q^{(i)} = \langle p_i^q \rangle. \tag{9}$$

This possibility is under study now.

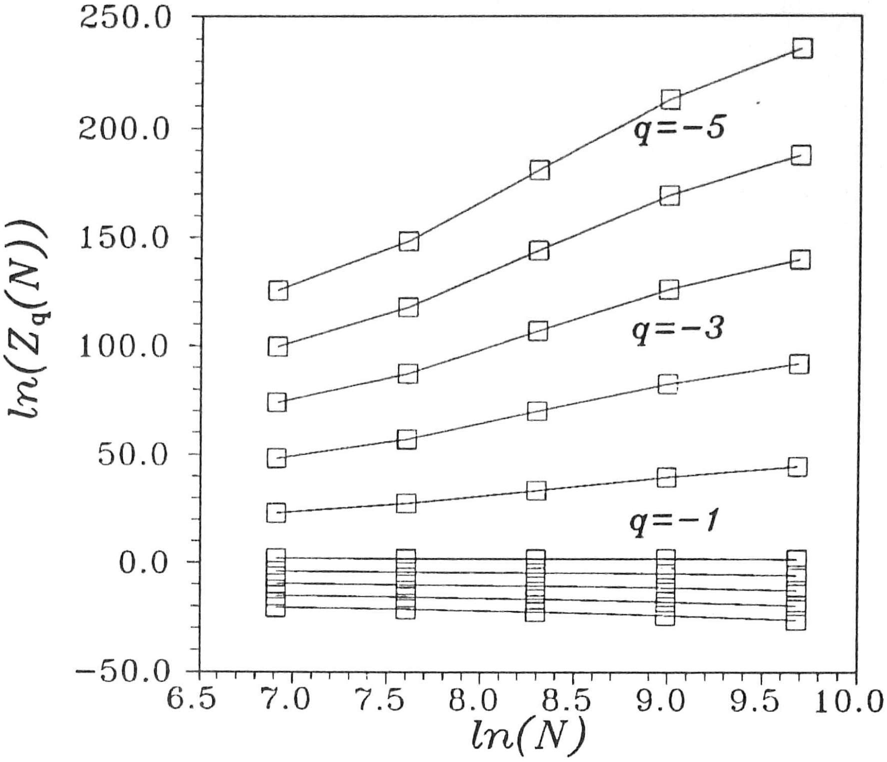


FIG. 1. The plots of $\ln < Z_q(N) >$ vs $\ln(N)$ for $q = -5, -4, \ldots 5$ with the exception of $q = 1$ (because $Z_q(N) = 1$) to test the scaling law (2).

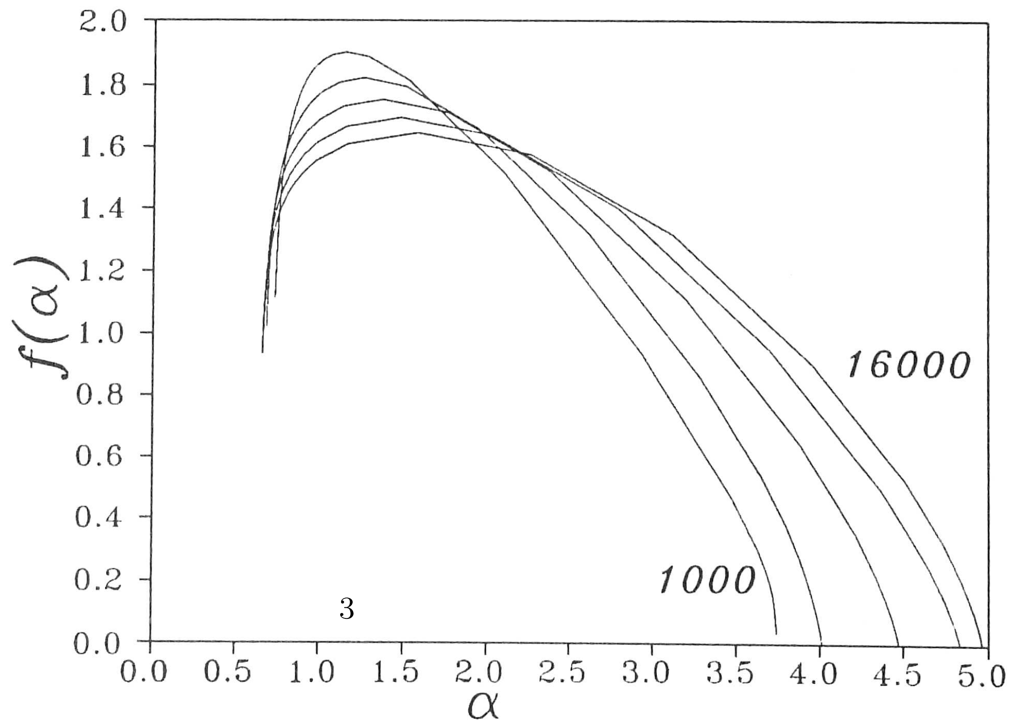


FIG. 2. The plots of $f(\alpha)$ for $N = 1000, \ldots, 16000$. The collapse of curves on the left-hand side of the maxima points and splitting of curves on the right-hand side is resembling the behaviour of multifractality spectrum obtained for DLA.

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