Generating function for Hermite polynomials

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The usual Hermite polynomials $H_n(x)$ are defined by the Rodrigues formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right).$$

The generating functions for polynomials $H_n(x)$ is

$$G(x,t) = \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!} \tag{1}$$

It is possible to obtain closed form for $G(x,t)$. Using the Cauchy's differentiation formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

where C is a contour enclosing the point z_0 and the integral is taken counter-clockwise, we calculate

$$\sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} e^{x^2} \frac{n!}{2\pi i} \oint_C \frac{e^{-z^2}}{(z-x)^{n+1}} dz = e^{x^2} \frac{1}{2\pi i} \oint_C e^{-z^2} \left(\frac{1}{z-x} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(z-x)^n} \right) dz$$

Summing the geometrical series we obtain finally

$$G(x,t) = e^{x^2} \frac{1}{2\pi i} \oint_C \frac{e^{-z^2}}{z-x+t} dz = e^{x^2} e^{-(x-t)^2} = e^{2xt-t^2}. \tag{3}$$

Indeed, we have after the change of variables $y = x - t,\ d/dt = -d/dy$:

$$\left.\frac{\partial^n G(x,t)}{\partial t^n}\right|_{t=0} = e^{x^2} \left.\frac{\partial^n e^{-(x-t)^2}}{\partial t^n}\right|_{t=0} = e^{x^2} (-1)^n \left.\frac{d^n}{dy^n} e^{-y^2}\right|_{t=0} = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = H_n(x).$$