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Remarks on the Uemura plot for weakly interacting charged Bose fluids

Zygmunt M. Galasiewicz, Marek Wolf

Institute of Theoretical Physics, University of Wrocław, Pl.Maxa Borna 9, PL-50-204 Wrocław, Poland and Z.M.G. Institute of Low Temperatures and Structures Research, of the Polish Academy of Sciences, Wrocław, Poland

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Abstract

In the paper Enz and Galasiewicz [*Physica C 214 (1993) 239*] one finds, for weakly repulsive Bose fluids the $T_c - n_s$ relation which reproduces reasonably well the straight line part of "Uemura plot". In the formula for superfluid density $n_s = n_s(t_1,\ t \sim 1)$ the integrals were presented as power series with respect to $t_1 = t_1(t_c) \ll 1$, where dimensionless parameters $t_1,\ t,\ t_c$ describe interaction, temperature and critical temperature, respectively. Here, the integrals have been calculated numerically in a wider interval of parameters. Among new results we have found that, while for weak interaction $t_c > t_c^0$ (t_c^0 denotes critical temperature for free bosons), for the stronger one, like in superfluid helium 4, $t_c < t_c^0$. As concerns curves $T_c(n_s)$ we find that in the "straight line" part the slight bending upward is local and very soon begins a much faster descent, like on the Uemura plot for type II high-T_c superconductors. The lowest experimental data for carrier densities n_s and $m^* = 5m_e$ lead to $\lambda_L \sim 1000$ Å, $\xi \sim 1$ Å, $T_c \sim 360$ K.

In Ref. [1] the $T_c(n_s)$ relation for weakly interacting charged Bose fluids has been calculated in order to compare with Uemura experimental relations for high-T_c superconductors [2].

In 1955 Schafroth [3] showed that the charged non-interacting Bose gas (characterized by coherence length $\xi \to 0$, $m = 2m_e$) behaves like a London superconductor exhibiting the Meissner–Ochsenfeld effect. This effect is described by the London penetration depth $\lambda_L = n_s^{-1/2}(4\pi r_b)^{-\frac{1}{2}}$, where n_s denotes density of superfluid component (density of superconducting carriers), $r_b = e^2/2m_ec^2$ is named classical boson radius. In order to examine the Uemura relation for interacting Bose fluid the Bogoliubov model [4] of weakly interacting Bose fluid was considered. The model describes free elementary excitations with the Hamiltonian:

$$\hat{H} = \sum_{\mathbf{p}} \epsilon(p) b_{\mathbf{p}}^{+} b_{\mathbf{p}}, \tag{1}$$

$$\epsilon(p) = \sqrt{(up)^2 + (p^2/2m)^2}.$$

Here u denotes the velocity of sound in the Bose fluid

$$u = \frac{2\hbar}{m}\sqrt{\pi n a}, \qquad a \ll n^{-\frac{1}{3}}, \tag{2}$$

where a denotes the scattering length (proportional to the interaction, i.e. $U = U_0 \sim a$, see Ref. [5]), n is the density of bosons and $n^{-\frac{1}{3}}$ the average distance among bosons. (Formula (2) describes also the velocity of sound at 0 K for superfluid helium 4 for $a = 2.6$ Å and $n^{-\frac{1}{3}} = 3.6$ Å, see Ref. [6]).

Like in Ref. [1] we use the following notation:

$$mu^2 = \pi(2\hbar)^2 na/m \equiv k_B T_1, \quad t = T/T_c^0,$$

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$$t_1 = T_1/T_c^0,\ \ t_c \equiv T_c/T_c^0,$$
$$s = T_1/T = t_1/t,\ \ s_c = t_1/t_c, \tag{3}$$

where the upper index "0" refers to non-interacting bosons ($a = 0$).

From Eq. (3) we see that $t_1 \sim a$, i.e. t_1 and $s = t_1/t$ vanish for non-interacting bosons.

In Ref. [1] from thermodynamic relations one finds expressions for the density of excitations $n_{ex}(a,T)$, the density denoted by $n_{int}(a)$ describing at $T = 0$ non-condensate density which is different from zero due to interaction, and for density of condensate $n_c(a,T) = n_c(t_1,t)$. The square root of the density of condensate treated as the order parameter should vanish at $t = t_c$. The equation for the determination of the critical temperature t_c, of the form $n_c(t_1,t_c) = 0$ also gives the important relation $t_1 = t_1(t_c)$. Moreover from the flow properties the density of the normal component $n_n(a,T)$ was determined and, finally, the density of the superfluid component $n_s(a,T) = n_s(t_1,t) = n_s(t_c,t)$. This leads to the realization of the determination of the $T_c - n_s$ dependence.

The above mentioned densities are expressed by integrals denoted by $I(s)$ and $K(s)$ involving the Bose distribution function $(e^{sx} - 1)^{-1}$. The integrals $I(s), K(s)$ are presented [1] as expansions in powers of $s = t_1/t$. For t_1 one considers only the interaction interval where one could show that $t_c > t_c^0$. Surprisingly, the straight line part of the Uemura plot in the same accuracy was reproduced reasonably well. However, it was suggested [1] that in the distinction to the Uemura plot of type-II superconductors presenting downward-bent curves the curves for the family of interacting Bose fluids show upward bending.

Consideration of some experimental data for the density of carriers n_s as well as replacement of $m = 2m_e$ by the effective mass m^* gives a possibility for preliminary estimations of the order of magnitude for the penetration depth $\lambda_L \sim 1000$ Å, the correlation length $\xi \sim 1$ Å and the critical temperature $T_c \sim 360$ K.

The numerical global calculations of the above mentioned integrals as functions of $s = t_1/t_c$ allow one to get e.g. relations $t_1(t_c)$ and $n_s(t_c,t)$ in wider intervals of parameters than in the case of the series expansion. Namely, at stronger interaction represented by t_1 one can get $t_c < t_c^0$, which is observed in the case of superfluid helium 4. Moreover the slight bending upward of $n_s(t_c,t)$ turned out to be a local one and very soon turned into the downward bending characteristic for type-II high-T_c superconductors.

For the density of excitations we have (see notation in formula (3))

$$\frac{n_{ex}(t_1,t)}{n} = t^{\frac{3}{2}}\frac{I(t_1/t)}{I(0)},\quad \frac{n_{ex}^0}{n} = t^{\frac{3}{2}}, \tag{4}$$
$$n_{ex}(t_1,0) = n_{ex}^0(0) = 0,$$

where

$$I(s) = s^{\frac{3}{2}}\int_0^\infty \frac{x}{e^{sx}-1}\frac{dx}{\sqrt{1+\sqrt{1+x^2}}} \tag{5}$$

and $I(s) \geq 0$. From Eqs. (14) and (42) of Ref. [1] we have

$$\frac{n_{int}}{n} = \frac{\sqrt{2}}{3I(0)}t_1^{\frac{3}{2}} \approx 0.2t_1^{\frac{3}{2}}. \tag{6}$$

The density of condensate $n_c(t_1,t)$ is expressed by

$$\frac{n_c(t_1,t)}{n} = 1 - \frac{\sqrt{2}}{3I(0)}t_1^{\frac{3}{2}} - t^{\frac{3}{2}}\frac{I(t_1/t)}{I(0)}$$
$$= 1 - \frac{n_{int}}{n} - \frac{n_{ex}}{n},$$
$$\frac{n_c(t_1,0)}{n} = 1 - \frac{\sqrt{2}}{3I(0)}t_1^{\frac{3}{2}},\quad \frac{n_c^0(t)}{n} = 1 - t^{\frac{3}{2}}. \tag{7}$$

As we mentioned above the critical temperature t_c is determined by

$$n_c(t_1,t_c) = 0, \tag{8}$$

(see also Ref. [7]).

The integral $I(s)$ was calculated using the 8-points self-adaptive Newton-Cotes method and the Eq. (8) solved numerically to find the relation between t_1 and t_c. The resulting curve $t_c(t_1)$ is presented in Fig. 1.

The careful numerical analysis allows us also to make the statement, that n_c near t_c tends to zero like

$$n_c \sim |\,t - t_c\,|^{2\beta} \tag{8a}$$

with the critical index $\beta = 0.5$, like for the free boson system. Formula (8) and Fig. 1 show that by consideration of densities like n_c, n_s it is not possible to change the critical temperature t_c at constant interaction strength t_1.

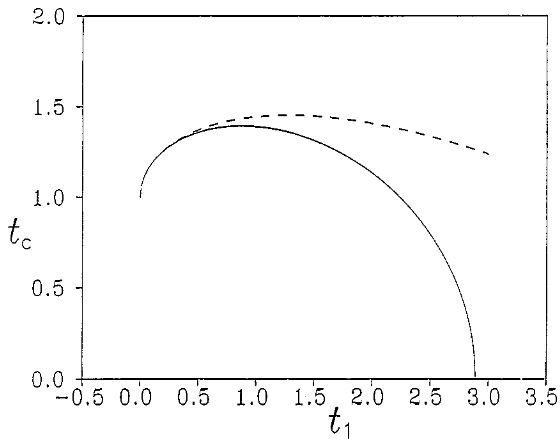

Fig. 1. The plot of the dependence $t_c(t_1)$ obtained as the solution of the Eq. (8). The dashed curve represents the analytical dependence obtained in Ref. [1].

Fig. 1 shows that for weak interaction $t_c > t_c^0 = 1$. For stronger interaction we have $t_c < t_c^0 = 1$. The latter situation is quite similar to that of liquid helium 4 (strong interaction). Namely, for the molecular mass of helium m_{He} and the density of the liquid state n_{He}, the critical temperature of an ideal Bose liquid is

$$T_c^0 = \frac{2^{\frac{1}{3}}\pi^{\frac{4}{3}}\hbar^2}{k_B I^{\frac{2}{3}}(0)}\frac{n_{He}^{\frac{2}{3}}}{m_{He}} = 3.13\text{K} \tag{9}$$

whereas

$$T_c = T_\lambda = 2.18\text{ K} < T_c^0. \tag{10}$$

In the case of determination of t_1 for superfluid helium 4 (see Eq. (3) and data for a and $n^{-1/3}$ given before this formula) one can use formally Eq. (6) presenting the fraction of the density of particles out of the condensate (at $T = 0$ K). With the help of Eq. (9) we have $t_1 = 2.69$ and

$$\frac{n_{int}}{n} = 0.88 = 88\%, \quad \frac{n_c(0)}{n} = 12\%. \tag{11}$$

which is close to the data given in Ref. [7]. Although the model considered here is applicable only to the weak interacting systems, the values obtained for helium 4 are not very bad. Namely, from the temperatures (9) and (10) we get $t_c = 0.696...$ and from the dependence $t_c(t_1)$ (see Fig. 1) we found the corresponding interaction parameter $t_1 = 2.6$ and finally Eq. (6) gives $n_{int}/n = 84\%$ — only a few percentages below experimental data. On the other hand, $t_1 = 2.69$ corresponds (Fig. 1) to $t_c = 0.58$ which gives $T_\lambda = t_c T_c^0 = 1.82$ K.

Let us also stress that the region of the maximum of the plot $t_c(t_1)$ is quite broad, i.e. the large critical temperature around maximum is attained for a broad range of interactions around the value $t_1^{(max)} \approx 0.87$ (the corresponding critical temperature is $t_c^{(max)} \approx 1.396$). Furthermore we mention that for $t_c > 1$ there are two values of t_1 being the solution of Eq. (8).

The density of the normal component n_n was determined [1] from the flow properties

$$\begin{aligned}\frac{n_n(t_1,t)}{n} &= t^{\frac{3}{2}}\frac{I(t_1/t)}{I(0)} - \frac{2}{3}\sqrt{t_1}t\frac{K(t_1/t)}{I(0)}\\ &= \frac{n_{ex}}{n} - \frac{2}{3}\sqrt{t_1}t\frac{K(t_1/t)}{I(0)}\\ \frac{n_n^0(t)}{n} &= t^{\frac{3}{2}} = \frac{n_{ex}^0(t)}{n}.\end{aligned} \tag{12}$$

Here the integral $K(s)$ is defined by the formula:

$$K(s) = s\int_0^\infty \frac{dx}{e^{sx}-1}\,\frac{\sqrt{\sqrt{x^2+1}-1}\,(\sqrt{x^2+1}+2)}{2(\sqrt{x^2+1})^3}. \tag{13}$$

and $K(s) \geq 0$. Formula (12) shows how the degeneracy ($n_n^0 = n_{ex}^0$) in case of noninteracting bosons is removed when the interaction and thereby t_1 is different from zero. Together with inequality $n_{ex} > n_n$ $(t \neq 0)$ this situation is presented in Fig. 2.

Having n_n given by Eq. (12) one gets from relation $n - n_n = n_s$ the expression for the density of superfluid component n_s

$$\begin{aligned}\frac{n_s(t_1,t)}{n} &= 1 - t\left\{\sqrt{t}\frac{I(t_1/t)}{I(0)} - \frac{2}{3}\sqrt{t_1}\frac{K(t_1/t)}{I(0)}\right\}\\ &= \frac{n_c(t_1/t)}{n} + \frac{\sqrt{2}}{3I(0)}t_1^{\frac{3}{2}} + \frac{2}{3}\sqrt{t_1}t\frac{K(t_1/t)}{I(0)},\end{aligned} \tag{14}$$

and because $K(s) \geq 0$ the inequality

$$\frac{n_s(t_1,t)}{n} \geq \frac{n_c(t_1/t)}{n}$$

is valid, see Fig. 3. Fig. 3 illustrates this situation in the form of $t_c - n_c$, $t_c - n_s$ relations. We see that in consequence the current density of condensate should

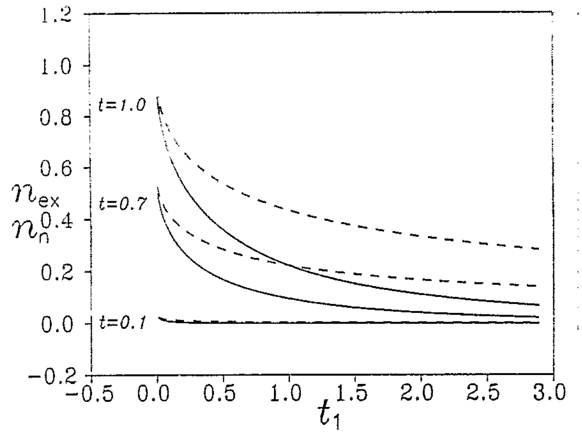

Fig. 2. An example of the inequality $\frac{n_{ex}}{n} \geq \frac{n_n}{n}$ for three values of the temperature $t = 0.1,\ 0.7,\ 1$. The curves n_{ex} are dashed and n_n are solid lines.

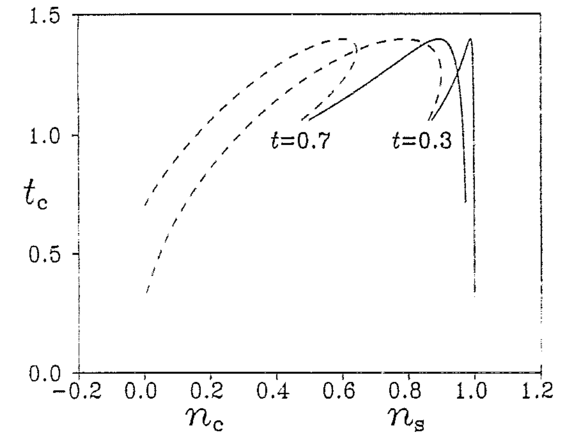

Fig. 3. This figure illutrates the inequality $\frac{n_s}{n} \geq \frac{n_c}{n}$ for two values of the temperature $t = 0.3,\ 0.7$. The curves $t_c(n_s)$ are drawn solid, and $t_c(n_c)$ are dashed. The gap between n_c (left) and n_s (right) is increasing with t_1. The curves are clipped in such a way, that only the physical region of the n_c is visible. For the sake of place the notation does not include denominators on the horizontal axis.

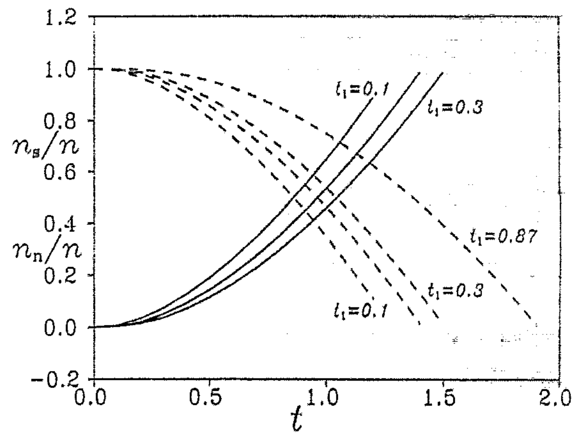

Fig. 4. This figure presents the temperature dependence of n_s and n_n as fraction of n for $t_1 = 0.1,\ 0.2,\ 0.3$. The plots of n_s/n are dashed. The additional curve for $t_1 = 0.87$ corresponds to the $n_s^{max}(t)$ for which t_c reaches maximum.

be smaller than the current density of the superfluid component: n_c is a fraction of n_s.

So, from Eqs. (12) and (14) we see that integral $K(t_1/t)$ describes the difference between n_n and n_{ex} as well as the difference between n_s and n_c. We have further the relations:

$$\frac{n_s(t_1,0)}{n} = 1 = \frac{n_c(t_1,0)}{n} + \frac{\sqrt{2}}{3I(0)} t_1^{\frac{3}{2}},$$
$$\frac{n_s^0(t)}{n} = 1 - t^{\frac{3}{2}} = \frac{n_c^0(t)}{n}. \qquad (15)$$

Formula (14) shows that the degeneration described by Eq. (15) is removed in the case when interaction and thereby t_1 is different from zero. From Eq. (8) it follows that n_c considered as the order parameter, vanishes exactly for $t = t_c$. On the other hand from Fig. 2 (together with Fig. 1) it follows that for $t \to t_c, n_r \to n$ $(r = ex, n)$, but $n_r(t_c) \neq n$. The reason is that the non-interacting quasi-particle description (1) like phonon–roton in HeII (see Ref. [5, 8]) is not valid in the vicinity of T_c. Fig. 4 presents values of n_s and n_n as fractions of the boson density n for different temperatures and interactions. As we will see below, the relation of n_s to n will be of special interest for us. From Fig. 4 it follows that for weak interaction and for $t \sim 1$ (i.e. $T \sim T_c^0$), $n_s \to 0$ and $n_n \to n$.

Fig. 5 presents the "Uemura plot", i.e. $t_c(n_s)$ dependence for weakly interacting Bose fluids. It reproduces the straight part of the plot with local slight upward tendency (i.e. Fig. 1 of Ref. [1]) but in the range of higher t_c for weakly interacting Bose fluids, like Uemura for high-T_c superconductors, we get a downward-bent curve. As it is seen from this figure, for any temperature t there is a particular value of $n_s^{max}(t)$ for which the t_c reaches its maximal value 1.396... — for $n_s > n_s^{max}$ ("overdoping" ?) the critical temperature begins to decrease. The plot of the quantity $n_s^{max}(t)$ versus t is also shown in Fig. 4 — with increasing t corresponding $n_s^{max}(t)$ decreases, which was to be expected.

As follows from the formula for λ_L (see in the text

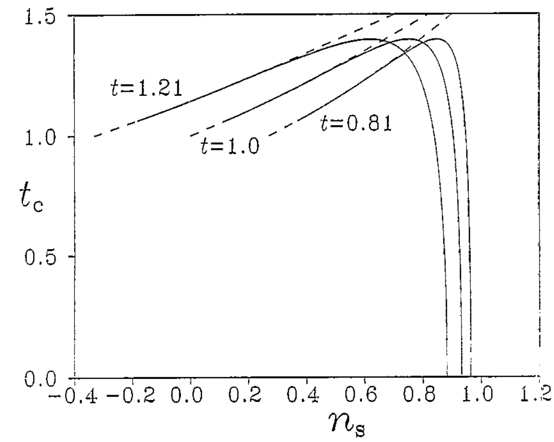

Fig. 5. The Uemura plots of the $t_c(n_s)$ dependence for three values of the temperature $t = 0.81,\ 1.0,\ 1.21$. These values of t are choosen to be the same as on Fig. 1 in Ref. [1]. The dashed lines represent the $t_c(n_s)$ dependence according to Eq. (50) from Ref. [1].

formula before formula (1)) it depends on the carrier density n_s and the boson mass. On the other hand the critical temperature (see formula (9)) depends on the boson density n and the boson mass m.

In Ref. [1] the boson mass has been taken to be $m = 2m_e$ and this choice has no influence on the shape of the curve $T_c - n_s$ we were interested in. However, a band structure and the renormalization effects lead actually to the effective mass $m^\star = m > 2m_e$. According to Ref. [10] we have e.g. for cuprate superconductors $2.4 \leq m^\star/m_e \leq 12 \pm 2.4$ (the upper limit is rather an exception) with the carrier densities within the limits $(0.00057 \pm 11)\text{Å}^{-3} - (0.0169 \pm 3.4)\text{Å}^{-3}$. As yet the highest ratio $m^\star/m_e = 23 \pm 5$ is for $TaS_2(Py)_{1/2}$. In Ref. [11] one finds that a typical value of the effective mass is $m^\star = 5m_e$. Formula (9) shows that too small masses and too high bason densities can lead to the nonrealistically high critical temperatures T_c^0 and T_c. (For weak coupling $t_1 \sim 0.2 \div 0.3$ formula (8) gives (see also Fig. 1) $t_c \sim 1.26 \div 1.3$, taking $t_c = 1.28$, we have $T_c = t_c T_c^0 = 1.28 T_c^0$).

Fig. 4 shows that for $t_1 \sim 0.1 \div 0.3$ $(t \sim 1)$ we have for $n_s = 0.5n$ (n_s is considered as the carrier density). We take the value for n_s from the lower limit of the experimental estimations of carrier densities from Ref. [11] e.g. $n_s = 0.0002\ \text{Å}^{-3} \rightarrow n = 0.0004\ \text{Å}^{-3}$ and take at the same time $m = 5m_e$. We get $\lambda_L \sim 1000\text{Å}$, $T_c = 1.28T_c^0 \sim 360K$. The formula (1.19) of Ref. [9] $\sqrt{\lambda_L/2m_e c} \leq \xi \leq \sqrt{2\lambda_L/2m_e c}$ gives the possibility to estimate the coherence length ξ when the penetration depth and the boson mass is known. For $\lambda_L \sim 1000\text{Å}$ and replacements $2m_e \rightarrow m^\star = 5m_e$ we have $\xi \sim 1$ Å.

For weakly interacting Bose fluids some important results were obtained on the basis of the here justified weak-coupling expansion (Ref. [1]). The crucial result is the determination of the straigth part of the Uemura plot. From the weak coupling data the relation $n_s \sim 0.5n$ was estimated. In the case of extension of calculation to larger values of the coupling constant the data obtained *formally* for ^{4}He, namely for the density of condensate at $T = 0$ and for T_λ are quite close to the experimental values. Finally, the Uemura plot $T_c(n_s)$ is reproduced in a proper way *beyond the power series expansion.*

The choice of the values for n_s and $m^\star$ from the experimental data leads to the quite reasonable values for λ_L, ξ and also for T_c.

For real bosons like ^{4}He atoms formula (9) describes the critical temperature. According to Ref. [10] the masses $m^\star$ of "complexes" which we consider as bosons are much smaller than $m_{He} \sim 4000m_e$ $(m^\star \ll m_{He})$. In this case in order to get from Eq. (9) reasonable critical temperature the boson density should be small. In consequence the carrier density $n_s \leq n$ should be small, like carrier density observed experimentally for HT_cS.

Finally it is worth to pay attention to the paper "Penetration Depth Measurements of 3D XY Critical Behaviour in $YBa_2Cu_3O_{6.45}$ Crystals" (Ref. [12]). In this paper the measured critical index for $1/\lambda_L^2(T) \sim (1 - \frac{T}{T_c})^{2y} \sim n_s \sim n_c$ is $y = 0.33$. This is consistent with the critical scaling in the universality class of the three dimensional XY model. It is also consistent with the critical behaviour of the density of the superfluid component n_s and the density of the condensate n_c for helium 4 (see e.g. Ref. [5], Section 28, and Sections 148,149 of Part 1). The consideration of the boson system in the present paper corresponds to a mean-field theory and therefore the exponent in (8a) is $\beta = 0.5$.

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