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FRACTAL STRUCTURES AND THEIR MEASURES

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1.Introduction

Two years ago at the previous School in this series of conferences I delivered a lecture *"Fractals. A short Introduction"* [1] and this talk is meant as a continuation of the previous one. In [1] I gave a few mathematical constructions of deterministic and random fractals and now I am going to discuss a few more realistic examples. In particular in Sect.2 I will introduce the Diffusion-Limited Aggregation model with which I will illustrate the notion of fractal dimension, correlation functions and the multfractal spectrum $f\alpha)$ in Sect.3.

Fractals have found many of applications in a large number of scientific disciplines. For the audience sitting in this lecture hall of most interest are applications of the fractal concept in biology and in this respect I would like to mention the review by Bruce J. West [2] (see also [3]) devoted in particular to the mathematical theory of the bronchial tree. It is a well known fact that our lungs in spite of a volume of about only a few liters possess an active surface of about $80 - 100\ m^2$. Such a large ratio of surface to volume is a generic property of fractals and for a mathematical constructions like the Koch snowflake (see Fig.1) the ratio of the length of its border to its area is infinity. It is unlike ordinary geometrical shapes such as, say, a circle or a ball of radius r, for which the ratio of length to the surface or surface to the volume diminishes as r^{-1}. There exist numerous other fractal-like structures in biology — for example, the bile duct system, the urinary collecting tubes in the kidney, the subcellular membranes in liver cells, the capillary beds, veins of leaves, the convoluted surface of the brain, the lining of the bowel. The similiarity of the retinal neurons to the Diffusion-Limited Aggregation (DLA) model was discussed by Frank Caserta *et al* [4]. There are also applications of fractals in geology [5]. In volcanic glass rock, olivine has formed crystal shapes, which are reminiscent of clusters grown by the DLA mechanism. The space distribution of galaxies is remarkably well approximated as a fractal with the fractal dimension $D_f = 1.23$, see [6,7]. For many pictures of fractal shapes appearing in Nature see the beautiful album [8].

2. Diffusion-Limited Aggregation as an archetyp for natural phenomena

As have mentioned in the previous section the patterns seen in Nature are often produced by random processes. Diffusion-Limited Aggregation was introduced by Witten and Sander [9] and a large literature devoted to this topic now exists [11]. The DLA is a simple stochastic model for the formation of clusters in any dimensional space. I will present

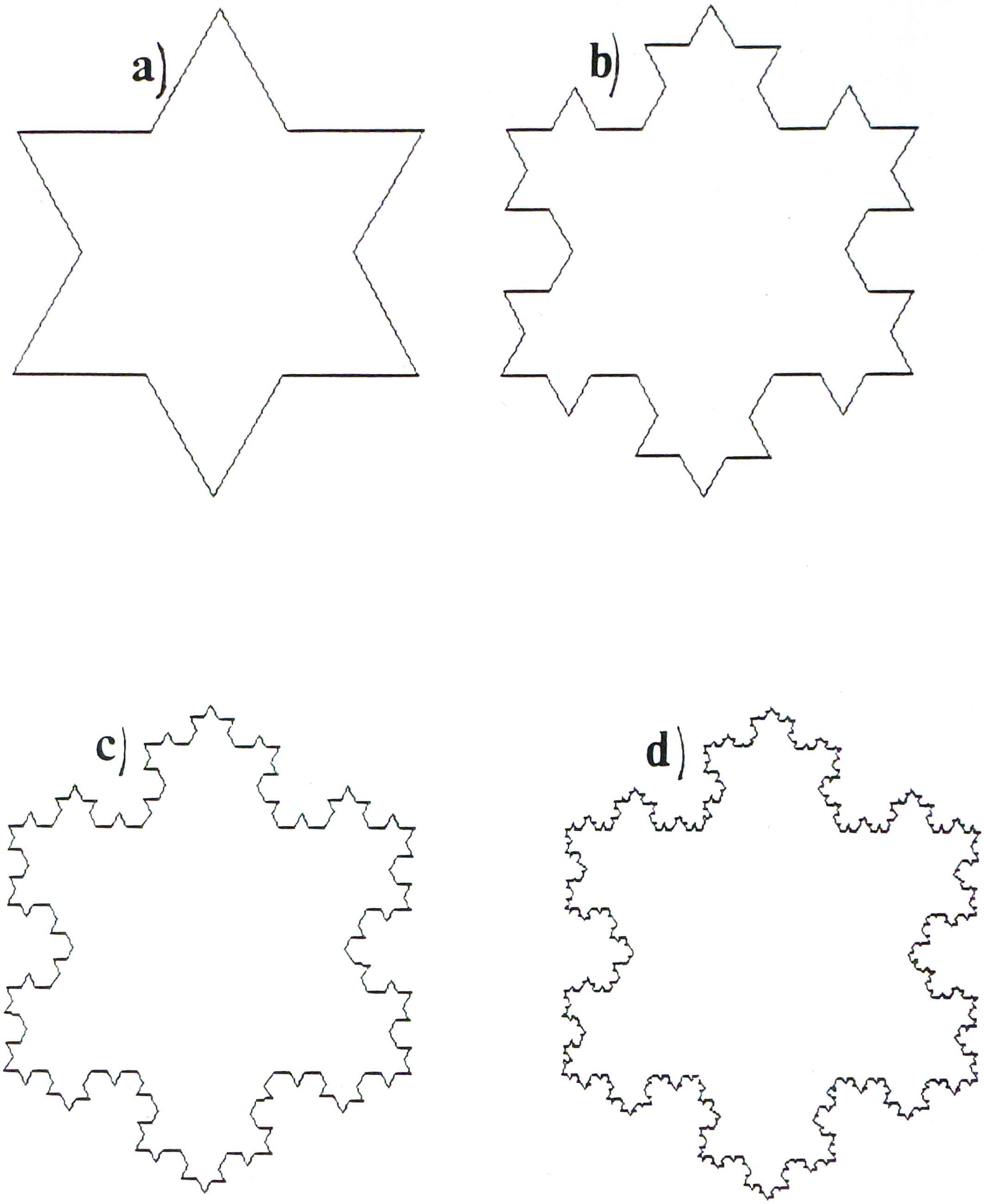

Fig.1 The Koch snowflake. The starting point is the equilateral triangle; it is a stage just preceeding the one shown in *a*). Each of the edges is replaced by the structure formed by four intervals of length equal to $\frac{1}{3}$ of the original one arranged as shown in *a*). This procedure is repeted *ad infinitum* as shown in *b*)-*d*). At each stage the length of the border increases by $\frac{4}{3}$, so the length of it diverges to ∞ as the geometrical progression $(\frac{4}{3})^{stage\ of\ iteration}$. In spite of the infinite length of the coastline the Koch snoflakes has a finite area as it can be encompassed by the square of the appriopriate size.

here the original [9] two dimensional version; the generalization two higher dimensions was introduced by Paul Meakin in [11]. In this model a single particle walks randomly on a square lattice until it reaches another particle (the "seed"), usually located at the centre of the lattice (there are many variants of this simple rule, also off-lattice). Then, a new particle initiates its random walk. If the particle contacts the cluster (now built of two particles) it is incorporated into the cluster and the cluster grows. This process is repeated many times ($10^3 - 10^6$) and leads to the formation of ramified clusters like the one shown in Fig.2. Many modifications and different techniques for computer simulations have been discussed, see [11]. Besides this "computer phenomenology" a number of real experiments have also been performed, such as electrodeposition [12], electric breakdown in dielectrics [13], fluid flows in porous media and viscous fingers [14]; see also [15]. As mentioned in the previous section DLA clusters are also similar to the many patterns encountered in Nature by biologists. Because of this importance of DLA we will choose that one cluster from Fig.2 to illustrate different quantities serving as a measure for fractal structures.

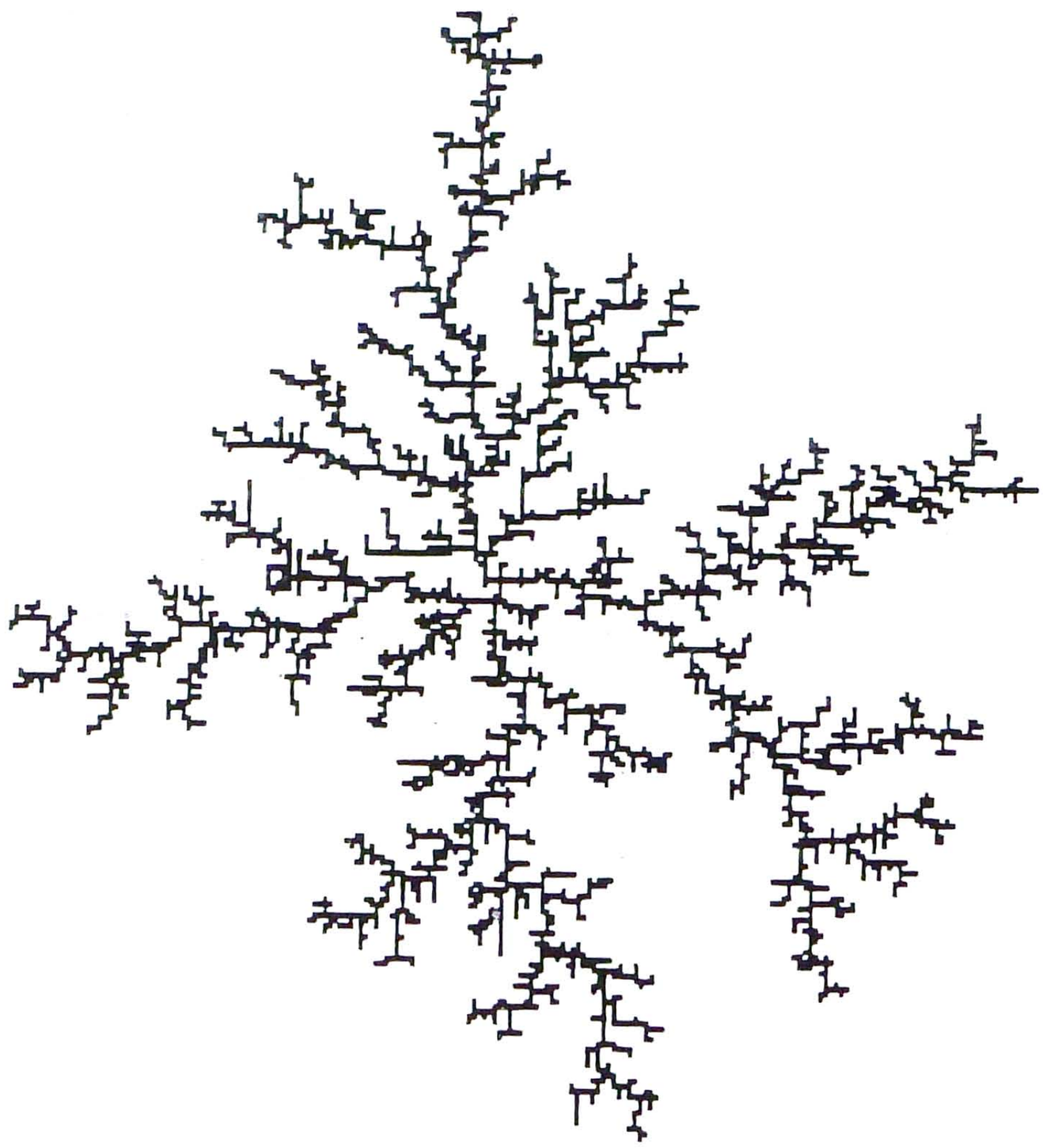

Fig.2 The DLA cluster formed from 4000 particles on the square lattice.

3.How to measure DLA clusters?

How to describe such a complicated structures as the strange Henon attractor shown on Fig.3 a-e in [1] or DLA cluster from Fig.2 ? There are a few numbers, analogous to the critical indices in the theory of phase transitions, used to characterize fractals. First of all there is a notion of fractal dimension D_f [16], but there are known examples of fractals with almost the same value of D_f but obtained by means of quite different mechanisms. For example the fractal dimension of the beckbone of percolation cluster is almost the same as D_f for the Sierpinski gasket as well as in $d = 2$ and $d = 3$, see [17]. Besides that both DLA and percolation clusters in $d = 3$ have almost the same D_f, but they look quite different. In view of this facts additional quantities are needed to discriminate between fractal structures. To describe these differences the "chemical" dimension D_l was introduced in [18]. For DLA clusters one finds $D_f = D_l$, while for percolation clusters in $d = 3$ one has $D_l = \frac{D_f}{1.35}$. Apart from D_f the notion of "lacunarity" was introduced by Mandlebrot in [16]. I will not dwell here on these indicies but below I will present fractal dimension, two different correlations functions, the continuous family of fractal dimensions D_q and the multifractal spectrum $f(\alpha)$.

3.a Fractal Dimension

In the end of my talk two years ago I have given rather abstract definition of fractal dimension [1], p.201. Here the special case of it applied to DLA cluster will be given.

In early works on DLA it was recognized, that the number of particles $N(r)$ inside the circle of radius r scales as a power:

$$N(r) \sim r^{-D_f}. \tag{1}$$

This scaling law is illustrated in the Fig.3 for a cluster from Fig.2. A variant of this relation is provided by the relation between the radius of gyration R_N and N. The radius of gyration is defined as follows:

$$\mathbf{R}_N \equiv \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=i+1}^{N} (\mathbf{r}_i - \mathbf{r}_j)^2}. \tag{2}$$

It is an easy calculation to show that $\mathbf{R}_N$ is identical to the mean-square average distance of the sites from the center of mass of the cluster:

$$\mathbf{R}_N = \sqrt{\frac{1}{N} \sum_i (\mathbf{r}_i - \mathbf{r}_{c.m.})^2}$$

$$\mathbf{r}_{c.m.} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{r}_i$$

For fractal structures $\mathbf{R}_N$ depends on N in the power-like manner:

$$\mathbf{R}_N \sim N^{1/D_f} \tag{3}$$

The log-log plot of this dependence is shown in Fig.4 and it is seen, that (3) holds with a great accuracy. Sometimes effective dimension D_{eff} defined as [19]:

$$D_{eff} = \ln 4/\ln\{(R_{2N}^2 - \frac{1}{4})/R_N^2\}$$

is used instead of (3) to get better results in the extrapolation.

3.b Correlation functions

There are two different functions named correlation functions what causes sometimes misunderstandings. The first one was introduced by Witten and Sander in [9] as the following average over different clusters (the outer bracket) and all solid angles (the inner bracket)

$$C(r) \equiv \langle \frac{1}{N}\langle\sum \rho(\mathbf{r}')\rho(\mathbf{r}+\mathbf{r}')\rangle_{\Omega(r)}\rangle \tag{4}$$

Here $\rho(\mathbf{r}) = 1$ if there is a particle at site $\mathbf{r}$; otherwise it is zero. Due to the averaging over angles the right hand-side in fact does not depend on $\mathbf{r}$ but on the radius $\mathbf{r}$ only. Taking into account that for empty sites $\rho(\mathbf{r}) = 0$ and performing the averaging over all $\mathbf{r}$ of a given length the above formula can be written in the simpler form as (we skip here the notation indicating the averaging over clusters)

$$C(r) = \frac{1}{2\pi N r}\sum_i n_i(r), \tag{5}$$

where $n_i(r)$ is the number of occupied sites on the circle of radius r around the site i. In this way the very time consuming summing over the lattice is replaced by the sum of the quantities $n_i(r)$ over particles . The plot of $C(r)$ for the DLA cluster from Fig.2 is shown in Fig.5. For large distances $C(r)$ decreases as

$$C(r) \sim r^{-\alpha} \tag{6}$$

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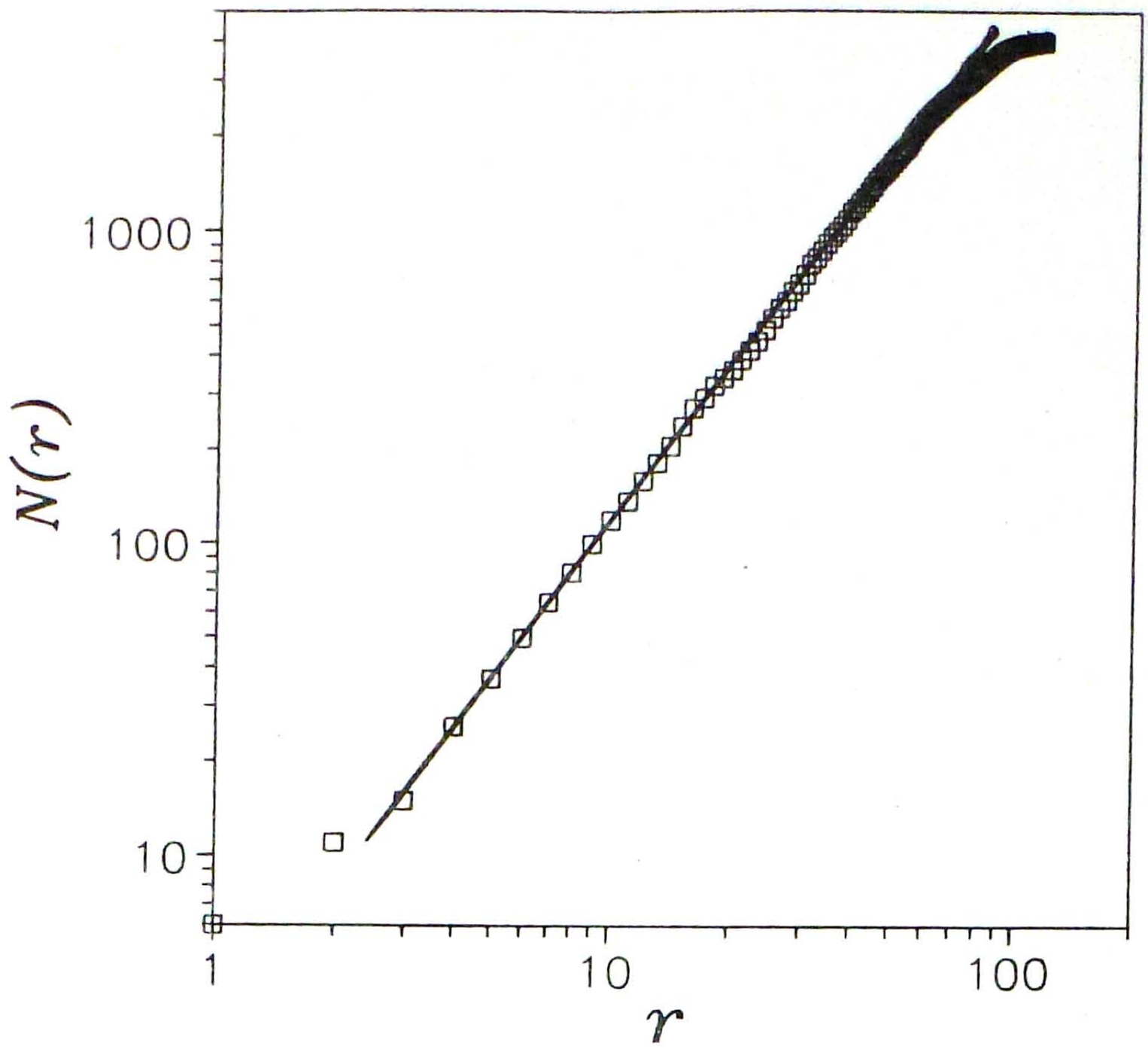

Fig.3 The plot of $N(r)$ vs r. The straight line has the equation $N(r) = 2.57r^{1.66}$.

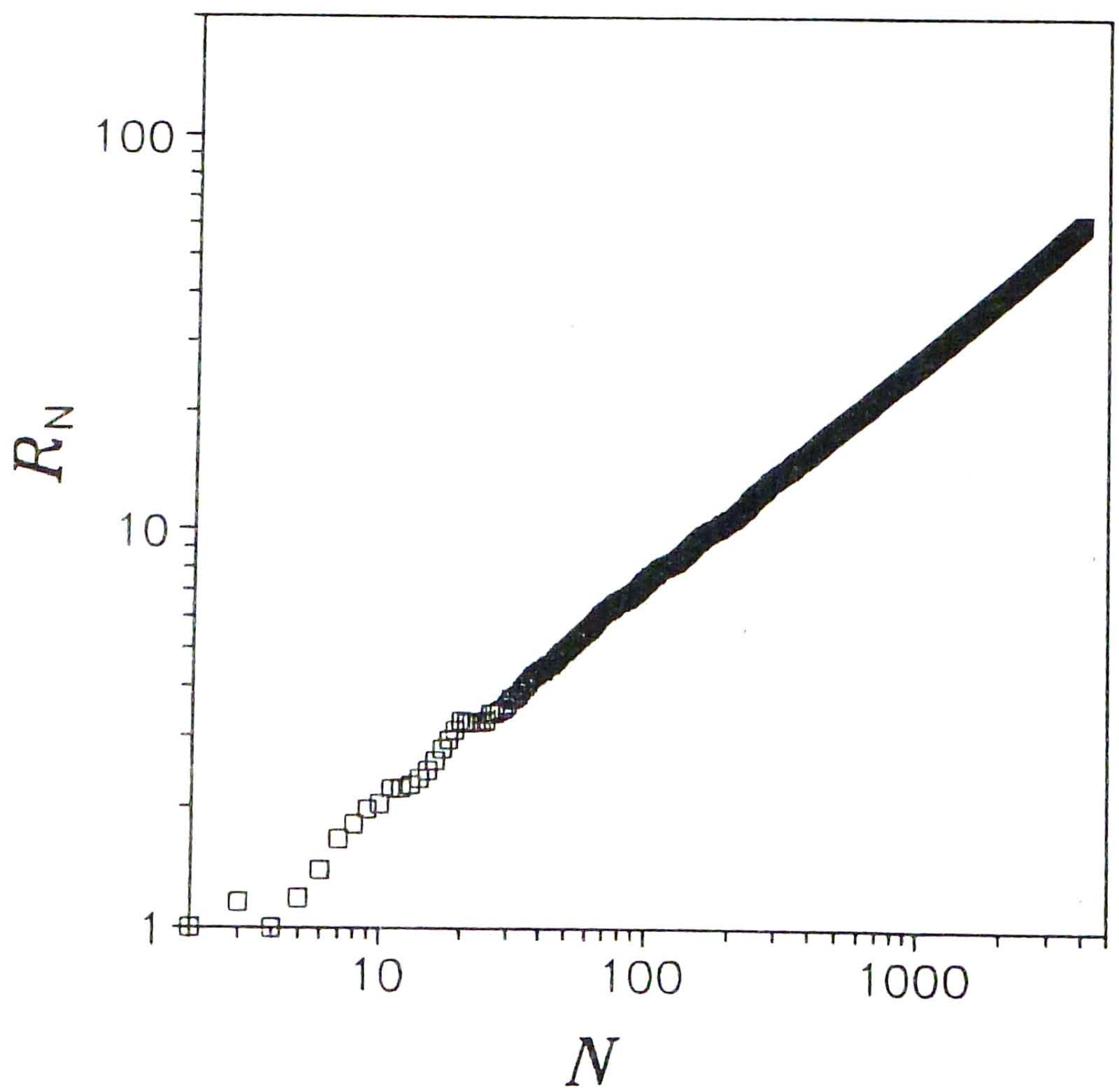

Fig.4 The plot of dependence of the radius of gyration for DLA cluster from Fig.2. The straight line has the equation $\mathbf{R}_N = 0.464N^{0.586}$. Let us remark, that 1/0.586=1.706 what is almost equal to the exponent in the dependence $N(r)$ on r.

where $\alpha = D_f - d$ in d - dimensional space. This relation between α and fractal dimension can be justified in the following way [20]. The discrete sum in the definition (4) can be rewriten in the continuum limit as follows:

$$C(r) = \frac{1}{N}\frac{1}{\int d\Omega_d(\hat{\mathbf{r}})}\int d\Omega_d(\hat{\mathbf{r}})\rho(\mathbf{r}')\rho(\mathbf{r}'+\mathbf{r}). \tag{7}$$

The radius of gyration can be rewritten in the continuum limit as

$$R_N^2 = \frac{1}{2N^2}\sum_{i,j}(\mathbf{r}_i - \mathbf{r}_j)^2 = \frac{1}{2N^2}\int d^d\mathbf{r}_1 \int d^d\mathbf{r}_2(\mathbf{r}_1 - \mathbf{r}_2)^2\rho(\mathbf{r}_1)\rho(\mathbf{r}_2). \tag{8}$$

Performing the change of variables $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}' = \mathbf{r}_2$ we get

$$R_N^2 = \frac{1}{2N^2}\int d^d\mathbf{r}\int d^d\mathbf{r}' r^2\rho(\mathbf{r})\rho(\mathbf{r}+\mathbf{r}'). \tag{9}$$

Using the identity $\int d^d\mathbf{r} \equiv \int dr r^{d-1}\int d\Omega_d(\hat{\mathbf{r}})$ and the formula (7) we obtain

$$R_N^2 = \frac{\Omega_d}{2N}\int dr r^{d+1}C(r), \tag{10}$$

where $\Omega_d = \Gamma(\frac{1}{2}d)/2\pi^{\frac{d}{2}}$ is the volume of the d dimensional unit sphere. Inserting here the dependences (3) and (6) we obtain the desired result $\alpha = D_f - d$. In fact there are corrections to this formula, see [20]. The generalization of (5) to higher dimensions takes the form:

$$C(r) = \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}Nr^{d-1}}\sum_i n_i(r) \tag{11}$$

Grassberger and Procaccia introduced another function called sometimes correlation integral [21]. They suggested simply to count how many pairs of points are there whose distance is less than r:

$$I_2(r) = \frac{2}{N(N-1)}\sum_{i=1}^{N}\sum_{j=i+1}^{N}\Theta(r - |\mathbf{r}_i - \mathbf{r}_j|) \tag{12}$$

Here $\Theta(x)$ is the Heaviside step function equal 1 when $x > 0$ and 0 for $x < 0$. In the appriopriate regime of r this function behaves as the power of r with exponent denoted by D_2 and called correlation dimension:

$$I_2(R) \sim r^{D_2} \tag{13}$$

The plot of this function for the DLA cluster from Fig.2 is given in the Fig.6. For the efficient algorithm of computing $I_2(R)$ see [22]. In fact D_2 is the special case of the more general definition of arbitrary q-order dimensions D_q given in [21] and next generalized to the multifractal spectrum $f(\alpha)$ in [23]; for the reviews see [24].

3.c Multifractal spectrum

The two point correlation integral $I_2(r)$ from equation (12) can be generalized to n-point function:

$$I_n(r) = \frac{1}{N^n}\{number\ of\ n-tuples\ of\ points(\mathbf{r}_{i_1}, \mathbf{r}_{i_2}, ...\mathbf{r}_{i_n})\ whose\ distances$$

$$|\mathbf{r}_{i_\alpha} - \mathbf{r}_{i_\beta}|\ are\ less\ then\ r\} \tag{14}$$

It can be shown that up to a factor of *O(1)* the above fromula can be rewritten as:

$$I_n(r) = \sum_i p_i^n(r) \tag{15}$$

where $p_i(r)$ is the probability to find the particle in the i-th box of the size r belonging to the covering of the cluster. Then it is found that $I_n(r)$ behaves as the power of r:

$$I_n(r) \sim r^{(n-1)D_n} \tag{16}$$

If the above procedure will be performed for the DLA cluster it will turn out that $D_n = const$, i.e. the exponents depend on n in the linear way: $(n-1)D_0$. The formula (15) has the further generalization to arbitrary real numbers:

$$I_q(r) = \sum_i p_i^q(r) \tag{17}$$

where now $q \in \Re$. If the function D_q depends on q the measure p_i is called multifractal. It can be shown that D_q is the decreasing function: $\frac{dD_q}{dq} < 0$. Next the Legendre'a transform of the function $\tau(q) \equiv (q-1)D_q$ is called the multifractal spectrum [23]

$$\alpha(q) = \frac{d\tau}{dq}, f(\alpha) = q\alpha(q) - \tau(q) \tag{18}$$

To obtain the nontrivial behavior of D_q for the DLA cluster we have to take as the measure p_i the probability for perimeter site i to be the next to grow [25].

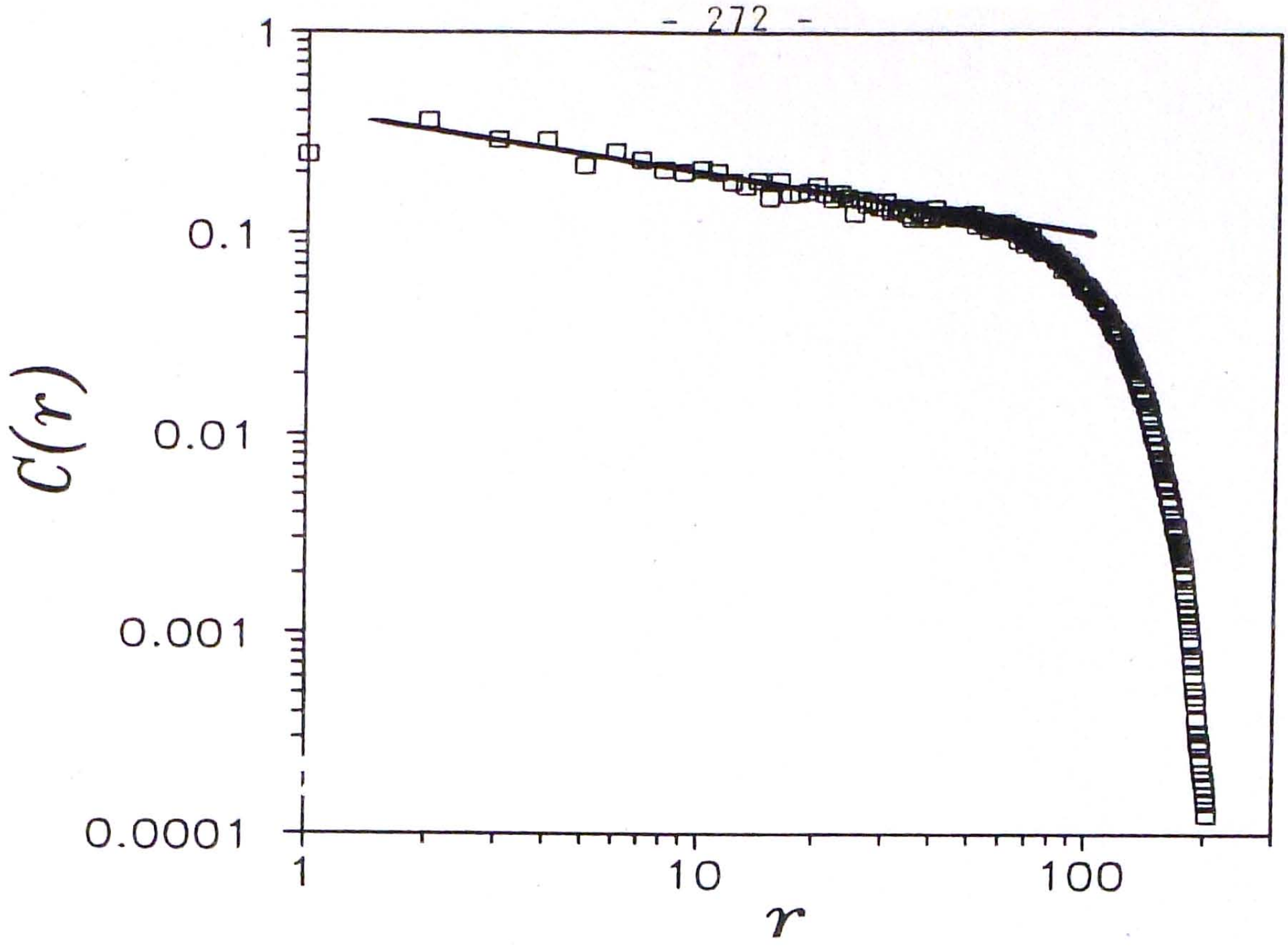

Fig.5 The plot of the correlation function for DLA cluster from Fig.2. The straight line has been fitted only to the points lying on the line - there is an abrupt decreasing of $C(r)$ for large r because the growth of the cluster was stopped and due to this fact $C(r)$ is smaller for large r then it should be if the growth would continue further. The equation of the straght line is $C(r) = 0.435r^{-0.34}$.

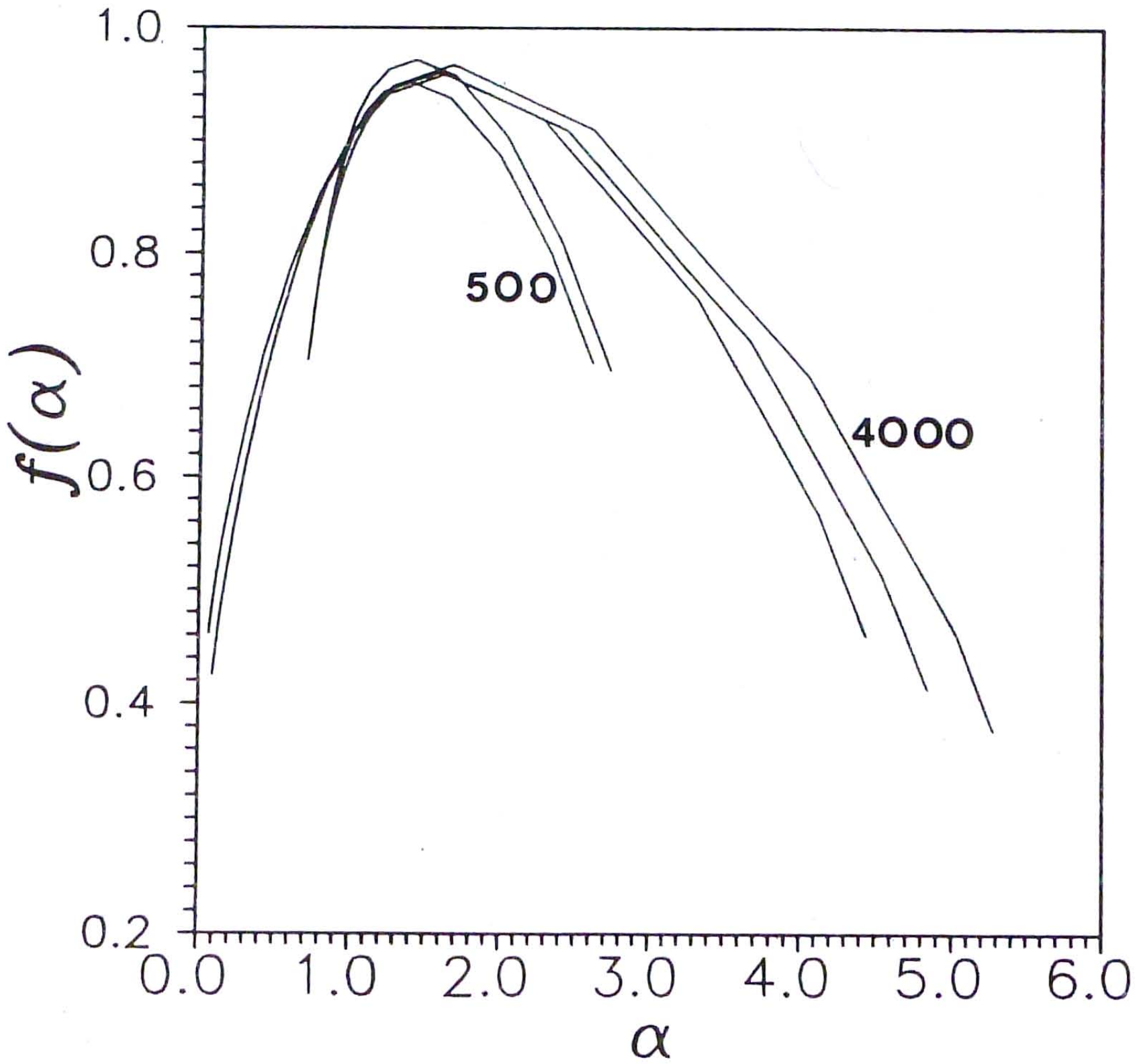

Fig.6 The plots of $f(\alpha)$ for the cluster from Fig.2.

There are a few methods of calculation the probabilities p_i. For such a large clusters as in Fig.2 I have used the method based on solving the Laplace equation, see [13]. In this electrostatic analogy of DLA the *surface* sites adjacent to the cluster and the enclosing boundary are held at different constant potentials ϕ_1 and ϕ_2, and the field ϕ satisfies the discrete Laplace equation $\nabla^2\phi_{i,j} = 0$ in the interior. The boundary conditions are $\phi = 0$ on the cluster and its *perimeter*. The probability $p_{i,j}$ that a particle will hit the site (i,j) is given by the formula

$$p_{i,j} = \frac{1}{4}(\phi_{i-1,j} + \phi_{i+1,j} + \phi_{i,j+1} + \phi_{i,j-1})$$

I have calculated $p_{i,j}$ at five stages of the growth process when the cluster consisted of 250, 500, 1000, 2000 and 4000 particles. The resulting functions $f(\alpha)$ are plotted on the Fig.6.

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