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ON THE CAT MAPPING AND FIBONACCI NUMBERS

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It is shown that the formula expressing the n-th iteration of the Cat Mapping by the initial one involves the elements of the Fibonacci sequence.

In this short note we are going to express the value of the n-th iteration of the Cat Mapping by the initial one. It turns out that this connection involves the even elements of the Fibonacci sequence. Fibonacci numbers are frequently met with the circle mapping, see e.g. [3, 4, 7].

Let us recall that the Cat Mapping transforms the two-dimensional torus into itself and is given by the formulas

$$x_{n+1} = x_n + y_n \bmod 1, \tag{1a}$$

$$y_{n+1} = x_n + 2y_n \bmod 1, \quad n = 0, 1, \dots \tag{1b}$$

The above equations define one of the most popular dynamical toy models, which belongs to the family of Anosov diffeomorphisms. It was invented by Arnold [1] as the illustration of many properties shared by dynamical systems. For example the transformations (1) are: area preserving, ergodic, mixing and they possess positive Kolmogorov entropy. The striking example of action of the transformation (1) is given by a series of pictures in the article by Crutchfield *et al.* [2]. In particular these figures show the mixing property of (1) as well as the periodicity of rational points under its action. (Only for irrational x, y the orbits are nonperiodic and fill out the whole torus.) The question of the existence of periodic orbits was recently related to the nineteenth century arithmetic by Vivaldi [8] and Percival and Vivaldi [6].

We are going to show the connection between Fibonacci numbers and the Cat Mapping. First of all let us write (1a, b) in the matrix form:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = T \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \bmod 1, \tag{2}$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \tag{3}$$

The determinant of the above matrix is equal to 1: $\det T = 1$. This is the reason that the Cat Mapping is area preserving—an analog of the Liouville theorem holds.

Let us mention that for an integer a and arbitrary x the following identity holds:

$$(a(ax \bmod 1)) \bmod 1 = (a^2 x) \bmod 1. \tag{4}$$

Due to this identity we can iterate (2) and write it in the form

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = T^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \bmod 1. \tag{5}$$

The matrix T is symmetric and can be diagonalized by means of the following matrix A:

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+1/\sqrt{5}} & \sqrt{1-1/\sqrt{5}} \\ -\frac{1}{\varphi}\sqrt{1+\sqrt{5}} & \varphi\sqrt{1-1/\sqrt{5}} \end{bmatrix}, \tag{6}$$

$$A^T A = 1,$$

$$\det A = 1,$$

where φ is the Golden Mean: $\varphi = \frac{1}{2}(1+\sqrt{5})$. The eigenvalues of T are $\lambda_1 = \frac{1}{2}(3+\sqrt{5})$, $\lambda_2 = 1/\lambda_1$, so the formula (5) can be written in the following form:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A \begin{pmatrix} \lambda_2^n & 0 \\ 0 & \lambda_1^n \end{pmatrix} A^T \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \bmod 1. \tag{7}$$

Using the explicit form (6) of the matrix A the following expression for (x_n, y_n) in terms of (x_0, y_0) can be derived from the above formula:

$$x_n = \frac{1}{\sqrt{5}} \left[\left(\varphi \frac{1}{\lambda^n} + \frac{1}{\varphi} \lambda^n \right) x_0 + \left(\lambda^n - \frac{1}{\lambda^n} \right) y_0 \right] \bmod 1, \tag{8a}$$

$$y_n = \frac{1}{\sqrt{5}} \left[\left(\lambda^n - \frac{1}{\lambda^n} \right) x_0 + \left(\varphi \lambda^n + \frac{1}{\varphi} \frac{1}{\lambda^n} y_0 \right) \right] \bmod 1, \tag{8b}$$

where we denoted $\lambda = \lambda_2 = (3+\sqrt{5})/2 = 1+\varphi$. We are going to calculate the coefficients in front of x_0 and y_0 in the formulas (8a, b). Let us recall that λ satisfies the eigenvalue equation for the matrix (3):

$$\lambda^2 - 3\lambda + 1 = 0. \tag{9}$$

Using (9) we can calculate λ^3:

$$\lambda^3 = \lambda(3\lambda - 1) = 8\lambda - 3.$$

It is easily seen that repeating the above trick we can express the n-th power of λ as the following linear combination:

$$\lambda^n = a_n \lambda + b_n. \tag{10}$$

Rewriting the above formula for $n+1$ and using (9), we obtain that the coefficients a_0 and b_0 are given by the following iterations:

$$a_{n+1} = 3a_n + b_n, \tag{11a}$$

$$b_{n+1} = -a_n, \tag{11b}$$

where initial values are $a_1 = 1$, $b_1 = 0$. Due to the relation (11b) the n-th power of λ can be expressed by the elements a_n alone:

$$\lambda^n = a_n \lambda - a_{n-1}, \tag{12}$$

where the sequence $\{a_n\}_{n=0}^{\infty}$ is given by the iteration

$$a_{n+1} = 3a_n - a_{n-1}, \; n \geqslant 1 \tag{13}$$

now with $a_0 = 0$, $a_1 = 1$.

Let us remark that as $1/\lambda$ is also a solution to the equation (9), the above reasoning can be applied *mutatis mutandis* to $1/\lambda$ giving the result

$$\frac{1}{\lambda^n} = a_n \frac{1}{\lambda} - a_{n-1}. \tag{14}$$

The selfconsistency of (12) and (14) leads to the following identity for a_n:

$$a_n^2 - 3a_n a_{n-1} + a_{n-1}^2 = 1. \tag{15}$$

In virtue of the formulas (12) and (14) the coefficients in front of x_0, y_0 in (8a, b) can be expressed by means of a_n and finally we obtain

$$x_n = [(a_n - a_{n-1}) x_0 + a_n y_0] \bmod 1, \tag{16a}$$

$$y_n = [a_n x_0 + (2a_n - a_{n-1}) y_0] \bmod 1. \tag{16b}$$

We can see now that the condition (15) expresses the fact that the determinant of the above transformation is equal to 1. It remains to be shown that elements of the sequence $\{a_n\}_{n=1}^{\infty}$ are equal to the even elements of the Fibonacci sequence $\{F_n\}_{n=1}^{\infty}$. Let us recall (see e.g. [5, § 1.2.8]) that the Fibonacci numbers are given by the following iteration:

$$F_{n+2} = F_{n+1} + F_n, \tag{17}$$

with the initial values $F_0 = 0$, $F_1 = 1$. It can be easily proved by the method of mathematical induction that F_{2n} fulfils the same recurrence formula as a_n does:

$$F_{2n+1} = 3F_{2n} - F_{2n-2}.$$

Using the initial conditions it follows that $a_n = F_{2n}$ and finally we can write

$$x_n = [(F_{2n} - F_{2n-2})x_0 + F_{2n}y_0] \bmod 1, \tag{18a}$$

$$y_n = [F_{2n}x_0 + (2F_{2n} - F_{2n-2})y_0] \bmod 1. \tag{18b}$$

Finally, using the definition (17) the formulas (18) can be written in the form

$$x_n = (F_{2n-1}x_0 + F_{2n}y_0) \bmod 1,$$

$$y_n = (F_{2n}x_0 + F_{2n+1}y_0) \bmod 1.$$

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BOOK REVIEWS

GREGORY L. BAKER (Academy of the New Church College, Bryn Athy Pennsylvania), JERRY P. GOLLUB (Haverford College, Haverford, Pennsylvani Chaotic Dynamics: an Introduction, Cambridge University Press, Cambridge, Ne York, Port Chester, Melbourne, Sydney, 1990. X + 182 pp. ISBN 0-521-38258-0 (H/ £25.00, $49,50) ISBN 0-521-38897-X (P/b, £9.95, $17.95)

The book *Chaotic Dynamics: an Introduction* by G. L. Baker and J. P. Gollub an intelligent presentation of the most important ideas in the theory of nonline dynamics and chaotic phenomena. Written with a junior undergraduate reader mind, the text appeals mainly to his physical intuitions, while precise mathemati technique is left aside, except for the necessary minimum. The prerequisites f studying the book are listed in the Preface: they include elementary multivarial calculus, linear differential equations and introductory physics. In addition, Chap 2 contains a brief presentation of basic mathematical ideas and notions such as pha space, Poincaré section, Fourier analysis of the time series etc.

It is the original idea of the authors to introduce all the aspects of chaotic moti through the detailed study of the driven damped pendulum,

$$ml\ddot{\theta} + \gamma\dot{\theta} + W\sin\theta = A\cos(\omega_D t).$$

Different regimes of the pendulum motion are carefully presented in a series computer-generated pictures including phase portraits, Poincaré sections, basins attraction, power spectra of the corresponding time series and bifurcation diagram

The book is supplemented with a software package available on a $5\frac{1}{4}$ in. diske at some extra cost (the complete source code of the programmes is given in A pendix B). The reader can use the software to run all the simulations himself, generate the graphics and the animated images of the pendulum in motion, to pl with parameters etc. Although we did not have a chance to test the software itself, would like to point out the novelty of the authors' idea to enrich the text w computer demonstrations and experiments to be run by the reader. Numero exercises at the end of each chapter refer to the software and suggest furtl simulations and experiments: the reader is expected to draw his own conclusio from the results and to discover some important facts by himself. The educatio value of such an approach is unquestionable.